an introduction to SETS

A. P. KEARNEY

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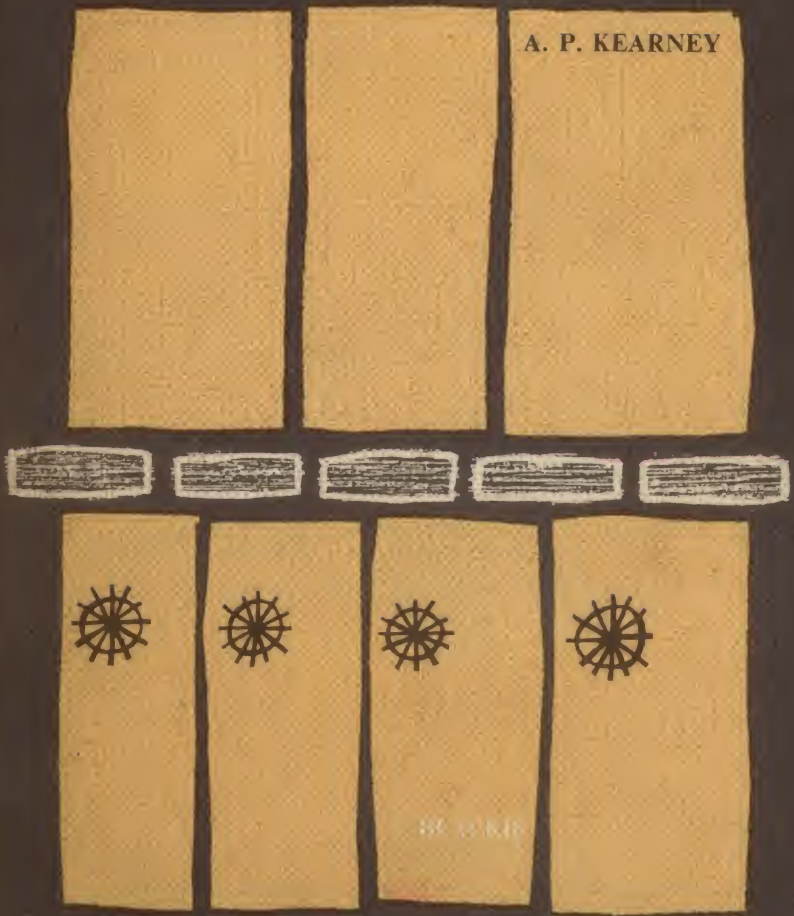

an introduction to SETS

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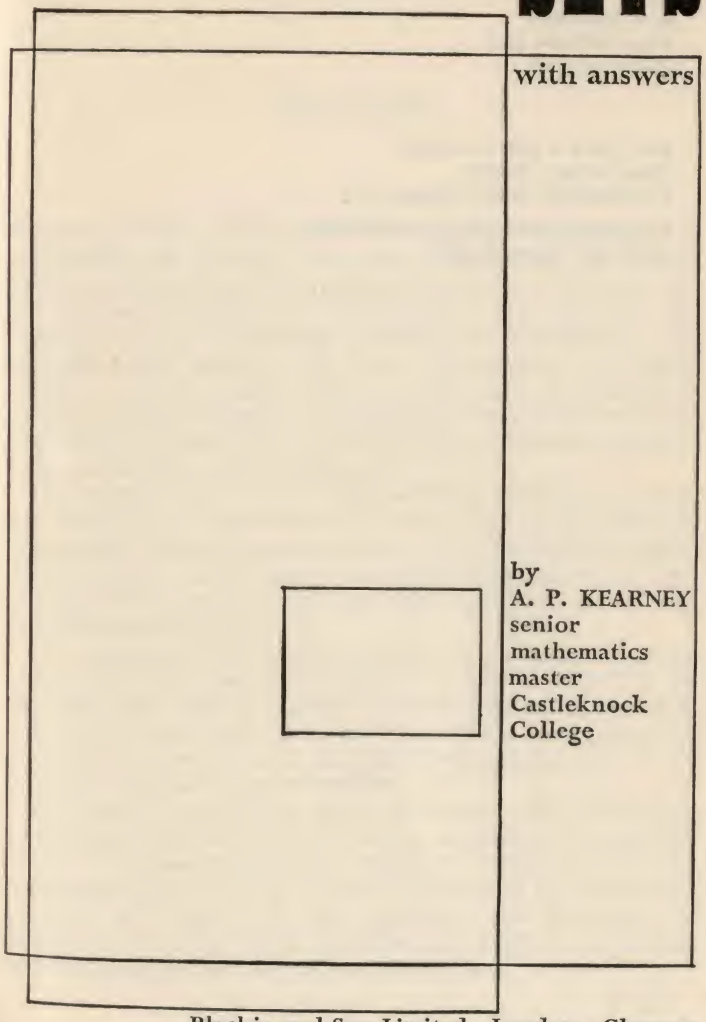
with answers



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Foreword

The new approach, on both sides of the Atlantic, to the teaching of mathematics has suggested the need of developing the number system leading students to an appreciation of the number line and the real number system; the introduction of the concept of the mathematical set, the language, notation, and symbolism of sets, especially sets of numbers; the idea of the function and relation viewed as sets of ordered points; a fuller study of inequalities, including graphical treatment of inequalities in one or two variables; and a more emphatic study of the basic laws of algebra, commutativity, associativity, and distributivity.

This modern approach has been embodied in the new syllabus prescribed by the Department of Education for Leaving Certificate Mathematics. It is hoped that this book will be of use to students in both the Pass and Honours Leaving Certificate Courses, and also to Intermediate students who are being introduced to the new thinking in mathematics.

The misgivings of the young student, ever on his guard against abstraction ("But what use is this, sir?") may be allayed by the assurance that Boolean algebra has very practical applications, not only in formal logic, but also in the design and construction of electronic computers, in dial telephone systems, and in the mysterious new subject of cybernetics.

The author is indebted to Sr. M. Benvenuta, O.P., Dominican College, Sion Hill, Blackrock, for advice in preparing the material; to Rev. Brendan Kearney, S.J., Belvedere College, Dublin, who read the proofs; and to Mr. John D. Sheridan of the Educational Company of Ireland, Ltd., for suggestions and guidance. For himself, on the other hand, he claims full credit for any defects and shortcomings that may be found in the text.

A.P.K.

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CHAPTER I

Sets and Subsets

In everyday language, a "set" is simply a collection of objects. Thus we speak of "a set of instruments", "a set of teeth", "a set of china".

The objects of which these sets are composed, however, are not clearly defined; thus the set of instruments may belong to a schoolboy or a dentist, and some instruments may be missing; the set of teeth may consist of 32, 29, or 27 individual teeth; and the set of china may be large or small.

In mathematics, however, a *set* is a collection of well-defined objects, and to make it clear that we are speaking of a mathematical set, we make use of *chain brackets* or *braces*.

Here are some mathematical sets:

A. {1, 2}

B. {All the rivers in Europe}

C. {All the boys in this school}

The objects of which a set is composed are called *elements* or *members*.

Thus 2 is an element (or member) of set A; the Danube is an element of set B; and each boy in the school is an element of set C.

If the number of elements or members of a set is small enough to be listed, we may write all the elements inside chain

brackets thus: {1, 3, 5, 7, 9}. This is the set of the first 5 positive odd numbers.

It is usual to use a capital letter to name a set. Thus we write: A = {1, 3, 5, 7, 9}. This is read: "A is the set of odd numbers greater than 0 and less than 10." For S = {0, 2, 4, 6, 8}, say "S is the set of even numbers from 0 to 8 inclusive".

NOTE: You may have a set with *no* members; like this, { }. This set, called for obvious reasons the EMPTY SET or the NULL SET, is of importance in mathematics, and is also indicated by the symbol ∅.

∅ denotes the empty set

{ } denotes the empty set

The set of square circles = ∅ or { }

WARNING. Do not confuse the empty or null set ∅ with the set {0}. The set {0} is not empty; it contains a very definite whole number, namely 0. If a thermometer shows 0 degrees Fahrenheit, it does not mean that there is NO temperature; if a man is standing 0 feet above sea level, it does not mean that he is standing NOWHERE.

There are innumerable examples of the empty set ∅. Round squares are non-existent, so the set of round squares is the empty set ∅. The set of boys under 2 years of age in this class is the empty set ∅. So too is the set of the rivers of the world over 10,000 miles in length; the set of people over 10 feet high; and the set of whole numbers between $\frac{1}{2}$ and $\frac{3}{4}$.

Distinction between the Empty Set ∅ and the Set {0}

Three boys entered for an examination: George Goodman, a worker; Tommy Talker, a drone; and Johnny Strong, another worker. Johnny Strong fell ill just before the examination and

was unable to sit. When the results appeared this is how they looked:

	Arith.	*Alg.*	*Geom.*	TOTAL
Goodman, G.	195	200	190	585
Talker, T.	0	0	0	0
Strong, J.	—	—	—	—

Goodman did excellently, the set of his marks being

{195, 200, 190}.

Strong's marks were non-existent; the set of his marks was ∅. Talker, as was to be expected, did badly, the set of his marks was {0, 0, 0} or more concisely {0}.

Needless to say, the set of compliments and congratulatory remarks offered to Tommy Talker by his father and mother was the empty set.

Moral: Anybody whose marks are {0} can expect ∅ of congratulations.

Subsets

The set of boys in one classroom of this school form a SUBSET of the set of boys in the whole school. Again, the set of boys in the last desk form a subset of the set of boys in the room.

{*a*, *b*, *c*} is a subset of {*a*, *b*, *c*, *d*}, because each of the 3 elements in {*a*, *b*, *c*} is an element of {*a*, *b*, *c*, *d*}. Similarly, {*a*, *b*, *c*, *d*} is a subset of {*a*, *d*, *m*, *c*, *b*}.

NOTE: {*a*, *c*, *d*, *b*, } is the same set as {*b*, *a*, *d*, *c*}, as the order in which the elements or members are set down is immaterial.

To indicate that one set is included in, or is a subset of another set, we use the symbol ⊂. Thus {*a*, *b*, *c*} ⊂ {*a*, *c*, *b*, *d*},

and $\{a, c, b, d\} \subset \{a, b, c, d, e\}$. These two statements could have been written more briefly thus:

$$\{a, b, c\} \subset \{a, c, b, d\} \subset \{a, b, c, d, e\}$$

If we let S denote the set of boys in the school,
C denote the set of boys in this classroom,
and L the set of boys in the last desk,
we may write $L \subset C \subset S$.

REMEMBER

The symbol $\subset$ means "is contained in" or "is a subset of". $\{1, 2\} \subset \{1, 2, 3\}$

In order that there should be no doubt about what elements or members are in a set, the elements may be either listed or described.

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10} is a set of ten members. The same set may be written {1, 2, 3, . . . 10}, the dots indicating intervening elements which can be deduced from the nature of the set.

Again, the same set is denoted by writing: "The set of positive integers (whole numbers) from 1 to 10, both included."

Proper Subsets

Out of a set of three boys whose names are A, B, and C we can select either 3 boys or 2 boys or 1 boy or 0 boy. These selections may be made in the following 8 ways:

1. {A, B, C}	**2.** {A, B}	**3.** {B, C}	**4.** {A, C}
5. {A}	**6.** {B}	**7.** {C}	**8.** { }

Set No. 8, in which no boy is picked, is the Empty Set or Null Set.

Notice that the list of 8 selections given above contains two extremes, (i) No. 1 which contains *all* the boys and (ii) No. 8 which contains *none* of them. Each of these, for purposes of uniformity, is regarded as a subset of the original set

{A, B, C}. Thus {A, B, C} is a subset of {A, B, C}. In other words, every set is a subset of itself; and the empty or null set { } is a subset of every set.

Sets 2 to 8, each of which has at least one element less than the original set, are known as *proper subsets.*

Thus if A is the set {1, 2, 3, 4, 5}, and B is the set {1, 2, 3, 4}, then B is a proper subset of A.

Definition of a Subset

Set A is a subset of set B if every element of set A is an element of set B.

Definition of a Proper Subset

Set A is a proper subset of set B if it is a subset which does not contain all the elements of B.

SUMMARY

A set is a collection of well-defined objects.

Each object in a set is called an element or a member of the set.

The elements of a set may be written within braces, thus: {1, 2, 3, 4}.

The symbol $\subset$ reads "is contained in" or "is a subset of".

Every set is a subset of itself.

Generally speaking, a proper subset is a subset which has fewer elements than the whole set.

The empty or null set, written { } or $\emptyset$, is a subset of every set.

EXERCISE 1

1. How many elements in the set {1, 2, 3}?
2. How many elements in the set {1, 1, 1, 2, 2, 3, 3, 3, 3}?
   (Remember: each element is counted once only.)
3. How many elements in the set {0, 1, 2}?

4. The set {1, 2, 3} has eight subsets. Write them.
5. The set {0, 1, 2} has eight subsets. Write them.
6. List the elements of the following sets:

   (*a*) All odd whole numbers from 1 to 21 inclusive.
   (*b*) All even whole numbers from 1 to 21 inclusive.
   (*c*) All the whole numbers from 0 to 50 inclusive which are perfect squares.
   (*d*) All the whole numbers from 10 to 30 inclusive which are multiples of 3.
   (*e*) All the whole numbers less than 20 and greater than 10.
   (*f*) All the whole numbers less than 10 and greater than 20.
   (*g*) All the towns of the world which have a population equal to or greater than 20,000,000.
   (*h*) All the whole numbers from 0 to 10, inclusive, which are the squares of whole numbers.
   (*i*) All the numbers from 0 to 10, inclusive, which are the square roots of whole numbers.

7. Describe, in words the following sets:

   (*Example.* {4, 6, 8, 10} is described as "The set of even numbers greater than 2 and less than 12".)

   (*a*) {3, 6, 9, 12, 15} (*d*) {4, 12, 20, 28, 8, 16, 24}
   (*b*) {10, 20, 30, 40, 50} (*e*) {1, 4, 9, 16, 25, 36, 49, 64, 81}
   (*c*) {4, 8, 12, 16, 20, 24, 28} (*f*) {5, 10, 15, 20, 25, 30, 35}

8. Given A = {10, 20, 30, 40, 50} and B = {5, 10, 15, 20, 25} write the set C containing all the elements common to A and B. (i) Is C a subset of A? (ii) Is C a subset of B? (iii) Is A a subset of B?

9. Use the symbol $\subset$ to write the following statement:
   If C is a subset of B, and B is a subset of A, then C is a subset of A.

CHAPTER II

Finite and Infinite Sets

A set is *finite* if its elements can be counted, with the counting coming to an end; otherwise the set is *infinite*. (The empty set is finite.)

The set of positive integers less than 50, that is {1, 2, 3, . . . 49}, is a finite set. It has 49 elements.

The set of positive integers greater than 50, that is

{51, 52, 53, 54, . . .}

is an infinite set; no matter how large a number we reach in counting, say, 100,000,029, we can always count further.

The phrase "infinitely great" is used colloquially to denote very large numbers, like the number of grains of sand in a desert. In mathematics, however, such numbers are NOT infinite.

EXERCISE 2

Classify the following sets as finite or infinite:

1. The set of all multiples of 3.
2. The set of all multiplies of 3 from 0 to 33, inclusive.
3. The set of all counting numbers.
4. The set of the squares of all counting numbers.
5. The set of all counting numbers less than one million.
6. The set of all the inhabitants of Asia.

Equal Sets

Two sets are equal if every member of each set is a member of the other set.

Thus: Look at these sets.

$$A = \{1, 2, 3, 4, 5\}, \qquad B = \{1, 3, 5, 2, 4\}$$

Here every element of A is an element of B, and every element of B is an element of A; or $A \subset B$ and $B \subset A$; or $A = B$.

One-to-One Correspondence

Imagine an untutored savage who has no idea of numbers, who can neither name numbers nor write numbers. He wishes to discover whether he has more red beads or more blue beads. He can solve his problem by pairing a red with a blue, one by one. He places one blue beside one red, then another blue beside another red, and so on, and so on. . . . If he runs out of blues while there are reds left over, he knows that he has more reds than blues; if he runs out of reds while there are blues left over, he knows that he has more blues than reds; but if he succeeds in pairing reds and blues evenly he knows that he has the same number of each.

This simple method of counting, known as ONE-TO-ONE CORRESPONDENCE, is very useful in set theory, especially in considering infinite sets.

Equivalent Sets

Consider the two sets: $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$.

We can arrange a one-to-one correspondence between the elements of the two sets thus:

$$\begin{array}{c} A = \{1, 2, 3, 4\} \\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow \\ B = \{a, b, c, d\} \end{array}$$

Such sets are called *equivalent* sets.

Two sets are equivalent if the elements of each can be put into one-to-one correspondence with the elements of the other. The symbol to denote that set A is equivalent to set B is $\sim$.

Thus: $A \sim B$.

WARNING. Do not confuse *equal sets* with *equivalent sets.*

Equal sets have the *same* elements. The set of people living in Scandinavia is the same set as the set of people living in the 3 countries of Scandinavia: Denmark, Norway and Sweden.

Equivalent sets can be put into one-to-one correspondence by matching one element from one set with one from the other so that no element is left over unmatched. In short, there are the same number of elements in each of two equivalent sets.

Equal sets are equivalent.

Equivalent sets are not necessarily equal.

Set-Builder Notation

Set-builder notation is used to give a short exact description of a set. Instead of writing

$$S = \{\text{the set of all integers greater than } 12\}$$

we may write $S = \{x \mid x \text{ an integer and } x > 12\}$.

Read: "The set of all x such that x is an integer and x is greater than 12."

S = "$\{x$" is read "S is the set of all x".

The stroke $\mid$ is read "such that".

The set B containing all the integers greater than 12 and less than 100 may be written

$$B = \{x \mid x \text{ an integer, and } 12 < x < 100\}$$

$12 < x < 100$ is a short way of indicating that x is less than 100 and greater than 12, that is, that x lies between 12 and 100.

Set-Builder Notation

$S = \{x \mid x$ (some rule or formula describing $x\}$

For example, the set of students in this class may be described in set-builder notation as follows:

$A = \{x \mid x$ is a member of this class$\}$

Examples

Tabular or "Roster" form.

1. $S = \{7, 9, 11, 13\}$
2. $R = \{6, 9, 12, 15\}$
3. $P = \{-3, -2, -1, 0, 1, 2, 3\}$

Set-Builder notation.

1. $S = \{x \mid x$ an odd integer and $15 > x > 5\}$
2. $R = \{3x \mid x$ an integer and $6 > x > 1\}$
3. $P = \{x \mid x$ an integer and $3 \geqq x \geqq -3\}$

NOTE: The symbol $\geqq$, which is sometimes written $\geq$, means "greater than or equal to".

The symbol $\leqq$, which is sometimes written $\leq$, means "less than or equal to".

SUMMARY

A set is finite if its elements can be counted with the counting coming to an end; otherwise it is infinite. The empty set is regarded as finite.

Two sets A and B are equal if every element of A is an element of B, and every element of B is an element of A; or $A \subset B$ and $B \subset A$.

Two sets are equivalent if the elements of one can be put into one-to-one correspondence with the elements of the other. All equal sets are equivalent.

Set-builder notation is a brief, precise method of describing a set. Thus $S = \{x \mid x$ is (some rule or formula describing $x)\}$.

EXERCISE 3

Assuming that x represents a whole number, write each of the following sets in tabular form, and state whether it is finite or infinite:

1. $\{x \mid 7 < x < 13\}$
2. $\{x \mid 7 < x \leqq 13\}$
3. $\{x \mid 7 \leqq x \leqq 13\}$
4. $\{x \mid x > 0\}$
5. $\{x \mid x \leqq 0\}$
6. $\{5x \mid x > 0\}$
7. $\{5x \mid x \geqq 0\}$
8. $\{x \mid 4 > x > -4\}$
9. $\{x \mid 4 \geqq x \geqq -4\}$
10. $\{2x \mid x > 100\}$

Write each of the following sets in set-builder notation, and state whether it is finite or infinite:

11. The set of positive whole numbers less than 30 and divisible by 3.

    (NOTE: The whole number 0 is neither positive nor negative.)

12. The set of positive whole numbers greater than 9, less than 30, and divisible by 3.
13. The set of positive whole numbers less than 100 that are divisible by 5.

14. The set of whole numbers greater than 100 and divisible by 5.

15. The set of all positive odd numbers.

16. The set of all positive even numbers.

17. The set of all fractions of the form $\frac{1}{x}$ where x is a whole number greater than zero and less than 21.

18. The set of all whole numbers between $\frac{1}{2}$ and $\frac{3}{4}$.

19. The set of all whole numbers greater than 100 and less than 10.

20. $S = \{x \mid x \text{ an integer and } 4 > x \geqq 0\}$ has 16 subsets. Write them down.

CHAPTER III

$\in$, *the Symbol for "is an Element of"*

In the set $S = \{1, 3, 4, 5, 7\}$, the fact that 5 is an element of S is written $5 \in S$. Say "5 is an element of the set S"
or "5 is a member of the set S"
or "5 belongs to the set S".

In the set $R = \{2, 4, 6, 8\}$, the fact that 5 is NOT an element of set R is written $5 \notin R$.

Say "5 is not an element of the set R"
or "5 is not a member of the set R".

$S = \{x \mid x \text{ an integer}, x > 0\} = \{1, 2, 3, \ldots\}$
$R = \{x^2 \mid x \text{ an integer}\} = \{0, 1, 4, 9, 16 \ldots\}$

In these two sets:

$1 \in S$ and $1 \in R$. $2 \in S$ but $2 \notin R$.
$3 \in S$ but $3 \notin R$. $4 \in S$ and $4 \in R$.

If $A = \{1, 2\}$ and $B = \{1, 2, 3\}$
$A \subset B$, that is, A is a subset of B.

Then if $x \in A$, it follows that $x \in B$,
or $x \in A \Rightarrow x \in B$.

(The symbol $\Rightarrow$ is read "implies".)

If $C = \{1, 2, 3, 1966\}$, and $D = \{1, 2, 3, 4, 5\}$, then C is not a subset of D, because 1966 is an element of C but is not an element of D.

This may be written: $C \not\subset D \because 1966 \notin D$.
($\because$ = because.)

NOTE: $\subset$ means *is a subset of*
$\not\subset$ means *is NOT a subset of.*

If $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d\}$, then $A \not\subset B$ because $e \in A$ and $e \notin B$.

REMEMBER
$12 \in R$ means "12 is an element of set R".
$15 \notin R$ means "15 is NOT an element of set R".

The Universe or Universal Set U

The first set of numbers a child learns is the set of counting or natural numbers. Since the only numbers at his disposal are those in the set of positive integers, the set of positive integers—for him and for the time being—is called the *universal set* of numbers.

Later, when he learns something of negative and fractional numbers, the universal set is extended to include such numbers.

The terms *universe* and *universal set*, however, are not confined to numbers. A universe is the overall set from which we form subsets.

Thus the sets representing, respectively, squares and parallelograms, are subsets of the "universe" or universal set representing all quadrilaterals. Similarly, the set representing the oak trees in England is a subset of the "universe" or universal set representing *all* the trees in England.

Universe or Universal Set

By *universe* or *universal set* we mean the totality of elements under discussion; it is the overall set from which we form subsets.

CHAPTER IV

Venn Diagrams

A very simple way to illustrate working with sets is by means of sketches known as "Venn Diagrams".

The rectangle U represents the set of all the schoolboys in the "universe" (our "universe", in this case, being the world).

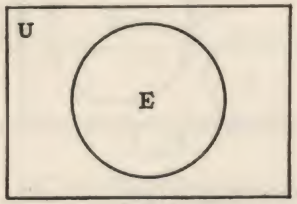


FIG. 1

The circle E represents the set of all the schoolboys in Europe.

The Venn diagram shows that $E \subset U$, because every member of E is a member of U.

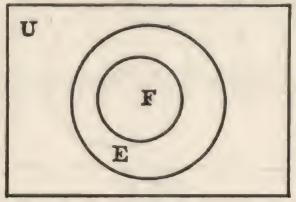


FIG. 2

In figure 2, the universe of schoolboys is represented by the rectangle U.

E represents the set of schoolboys in Europe, and F represents the set of schoolboys in France.

$F \subset E$ and $E \subset U$.

From the diagram it is obvious that if $F \subset E$ and $E \subset U$, then $F \subset U$.

Disjoint Sets

Set A represents the set of American-born boys, set E the set of European-born boys. Since no one boy can be both American-born and European-born, these sets have no common elements.

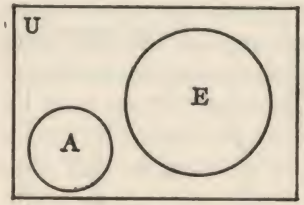


FIG. 3

Such sets are called *disjoint sets.*

European-born American-born people are non-existent. The set of American-born European-born boys, like the set of round squares, or the set of whole numbers between $\frac{1}{2}$ and $\frac{3}{4}$, is the empty set $\emptyset$.

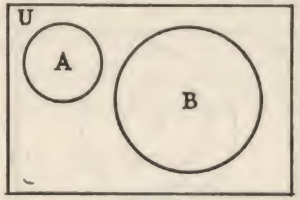


FIG. 4

If U = the set of all rational numbers, and A = the set of all odd integers, and B = the set of all even integers, figure 4 illustrates that A and B are disjoint sets, i.e. no element of

A is an element of B, and no element of B is an element of A.

A and B do not occupy the whole of U, since some rational numbers are not integers.

Complementation

If U = the set $\{1, 2, 3, 4, 5, 6\}$ and A its subset $\{2, 4, 6\}$, the remaining elements of U form the subset $\{1, 3, 5\}$.

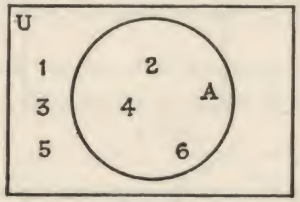


FIG. 5

The subset $\{1, 3, 5\}$ is called the COMPLEMENT of A, and is written A′ or $\bar{A}$.

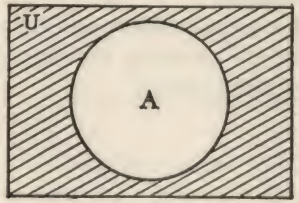


FIG. 6

Similarly, the shaded part of this diagram is the complement of the set A.

Definition

If A is a subset of a given universe U, then A′, the complement of A, is another subset containing all the elements of U which are not in A.

$$A' = \{x \mid x \notin A, \text{ but } x \in U\}$$

Partition of a Universe

Let A = set of all persons up to and including those 3 feet in height, and B = set of all persons more than 3 feet high. (The universe is the set of all people.)

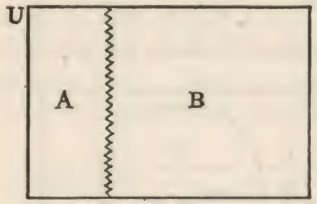


FIG. 7

The sets A and B are said to *partition* the set of all persons in U. In a partition there is no overlapping. Clearly A is the complement of B, and B is the complement of A.

Definition of Partition

If subsets of a given universe are disjoint and together make up the whole universe, the universe is said to be partitioned into these subsets.

CHAPTER V

Intersection $A \cap B$

Three Overlapping Circles

In this diagram, the set of all elements which are in both A and B, called *the intersection of A and B*, is represented by the shaded portion.

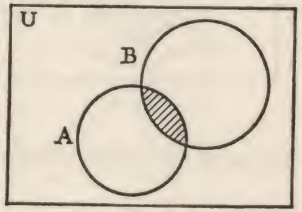


FIG. 8

(*a*) This shaded portion, the intersection of set A and set B, is written $A \cap B$.

If set A = set of all people with red hair, and set B = set of all bad-tempered people, then $A \cap B$ = set of all bad-tempered red-headed people.

(*b*) In figure 9, $A \cap B$ is the empty set $\emptyset$.

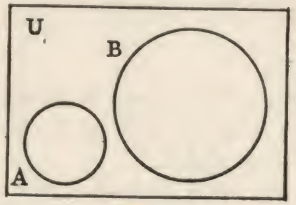


FIG. 9

If A and B denote the same sets as before, $A \cap B$ tells us that red-haired bad-tempered people are non-existent; in other words, that no red-haired person is bad-tempered.

$$A \cap B = \emptyset$$

(*c*) Figure 10 indicates that all red-haired people have bad tempers. (Also, that people without red hair *can* be bad-tempered.)

$$A \cap B = A$$

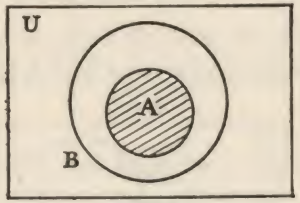


FIG. 10

(*d*) Figure 11 indicates that some red-haired people are *not* bad-tempered.

$$A \cap B = B$$

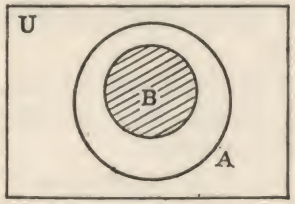


FIG. 11

Definition of $\cap$

The intersection of A and B, written $A \cap B$, and read *A intersection B*, or *A cap B*, is the set of elements which belong to both A and B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$
$$A \cap B \cap C = \{x \mid x \in A \text{ and } x \in B \text{ and } x \in C\}$$

If three overlapping circles are drawn showing the sets A, B, and C as subsets of a universal set U, eight disjoint sets, S_1, S_2, S_3, . . . S_8 arise.

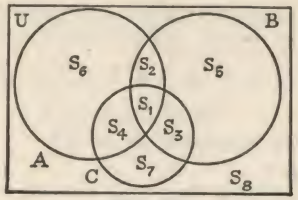


FIG. 12

$$S_1 = A \cap B \cap C$$
$$S_2 = A \cap B \cap C'$$
$$S_3 = A' \cap B \cap C$$
$$S_4 = A \cap B' \cap C$$
$$S_5 = A' \cap B \cap C'$$
$$S_6 = A \cap B' \cap C'$$
$$S_7 = A' \cap B' \cap C$$
$$S_8 = A' \cap B' \cap C'$$

The eight sets listed above are disjoint sets, that is, the intersection of any two gives the empty set $\emptyset$.

EXERCISE 4

1. Explain what is meant by the complement of A. How is the complement of A written?
2. Copy this Venn diagram and shade it to indicate A′.

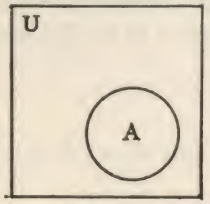


FIG. 13

3. Write in set-builder notation: The intersection of two sets A and B is another set C, which includes only the elements common to A and B.
4. Draw a Venn diagram in which the universe U is partitioned by sets A, B, C, D, E, and F.
5. Given $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 5\}$.
   (i) What is the complement of A?
   (ii) What is B′?
   (iii) Find $A \cap B$.
   (iv) Find $(A \cap B)'$, that is, the complement of $A \cap B$.
6. Given $U = \{x \mid x \text{ an integer}, 0 < x < 8\}$, and
   $$A = \{2x \mid 0 < x < 3, x \text{ an integer}\}$$
   write the elements in A′.
7. Given $U = \{x \mid x \text{ an integer, and } 18 > x > 7\}$,
   $A = \{x \mid x \text{ an integer and } 15 > x > 10\}$,
   and $B = \{x \mid x \text{ an integer and } 16 > x > 9\}$,
   write: (i) A′; (ii) B′; (iii) $A \cap B$; (iv) $(A \cap B)'$.
8. If $U = \{a, b, c, d, e, f\}$, $A = \{a, c, e\}$, $B = \{a, b, c, d\}$, find (i) $A \cap B$; (ii) $(A \cap B)'$; (iii) A′; (iv) B′.
9. Draw Venn diagrams to illustrate in a universe U, having subsets A and B, the following:
   (i) $A \cap B = \emptyset$; (ii) $A \cap B = A$; (iii) $A \cap B = B$.
10. A is a subset of the universe U. Draw Venn diagrams to show: (i) $A \cap U = A$; (ii) $A \cap A' = \emptyset$;
    (iii) Can we draw a diagram to show $A \cap \emptyset = \emptyset$?
11. If A = the set of all tall people, and B = the set of all dark people, interpret: (i) $A \cap B$; (ii) $(A \cap B)'$.

CHAPTER VI

Classification

In a certain town, sets of the men fall into the following categories (assumed well defined): T = the set of tall men; D = the set of dark men; and H = the set of handsome men. A person made the following table:

Sets and characteristics	Subsets included
(1) T	*a, b, c, d*
(2) D	*a, b, f, e*
(3) H	*a, d, e, g*
(4) T D	*a, b*
(5) D H	*a, e*
(6) H T	*a, d*
(7) T D H	*a*

It is required to interpret this table as a Venn diagram.

STEP I

Since there is a set $\{a\}$ representing the men who are tall, dark, and handsome, the three circles representing, respectively, the tall, the dark, and the handsome men, have a common area, the set $\{a\}$.

Draw these three circles, and mark in a. (See figure 14.)

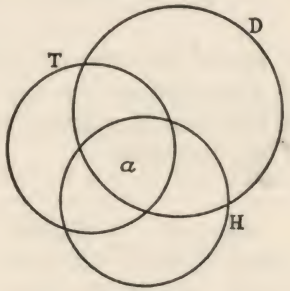


FIG. 14

STEP II

From the information in rows 4, 5, and 6, mark $\{d\}$, $\{b\}$, and $\{e\}$, as shown in figure 15.

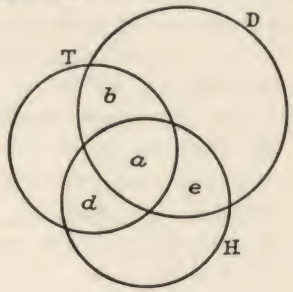


FIG. 15

STEP III

From the information in rows 1, 2, and 3, complete the Venn diagram as shown in figure 16.

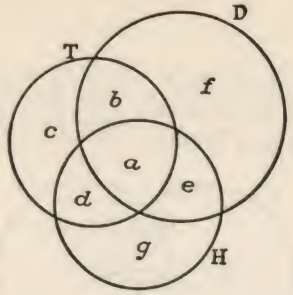


FIG. 16

Since in this discussion we are confining our observations to the tall, the dark, and handsome among men, there is no need to consider the universal set of all men, which includes men who are neither tall, nor dark, nor handsome.

Example

Out of 68 students in a school, 30 take French, 50 take Science, 24 take German; 22 take French and Science, 14 take Science and German, 10 take German and French; and 10 take all three subjects—French, German, and Science. Interpret this information in a Venn diagram.

STEP I

10 students take all three subjects.

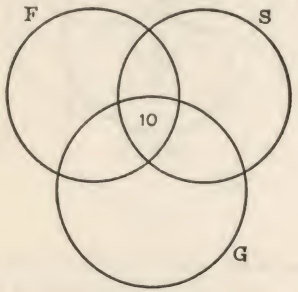


FIG. 17

STEP II

22 take French and Science $(10+12 = 22)$.
14 take German and Science $(10+4 = 14)$.
10 take German and French $(10+0 = 10)$.

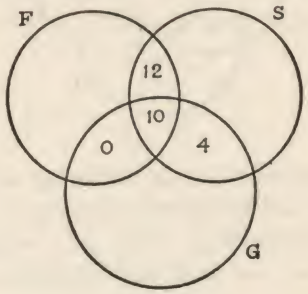


FIG. 18

STEP III

We now fill in the figures 8 (since 30 take French), 24 (since 50 take Science), and 10 (since 24 take German).

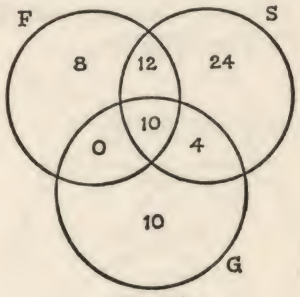


FIG. 19

Example

Out of 75 persons who were asked to write a list of what they normally drank (confining their preferences to coffee, tea, and

milk), 24 listed coffee, 43 listed tea, and 37 listed milk. Tea and coffee appeared on 7 lists, tea and milk on 13, and milk and coffee on 12.

Every list contained at least one of the beverages. How many lists contained all three?

STEP I

Suppose x lists contain all three.

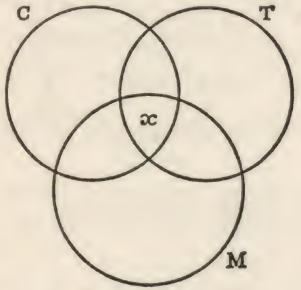


FIG. 20

STEP II

Since 12 lists contained milk and coffee, $12-x$ is put in the space shown by the arrow.

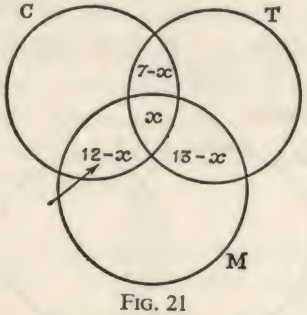


FIG. 21

Similarly for the other combinations.

STEP III

Since 24 listed coffee, $24-(7-x)-(12-x)-x$, or $5+x$, goes where indicated.

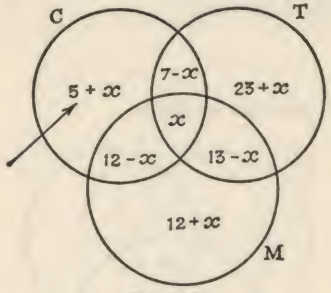


FIG. 22

Similarly for the other combinations.

STEP IV

$$5+x+7-x+23+x+13-x+12+x+12-x+x = 75$$
$$72+x = 75, \quad \therefore x = 3$$

Result: Three lists contained all three beverages.

SUMMARY

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$, or, the intersection of A and B, written $A \cap B$, is the set of elements which belong *both* to A and B.

$A \cap B$ is shaded.

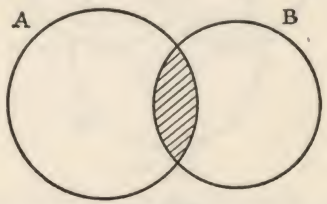


FIG. 23

EXERCISE 5

1. If F = set of students taking French, S = set of students taking Science, G = set of students taking Greek, interpret:

   (*a*) $F \cap S \cap G$; (*b*) $F' \cap S \cap G$; (*c*) $F \cap S' \cap G$;
   (*d*) $F \cap S \cap G'$; (*e*) $F' \cap S' \cap G$; (*f*) $F' \cap S \cap G'$;
   (*g*) $F \cap S' \cap G'$; (*h*) $F' \cap S' \cap G'$.

2. Make 8 separate Venn diagrams to illustrate the eight sets (*a*) to (*h*) in question 1. (See figure 12, page 31.)

   (NOTE: $n(A)$ denotes the number of elements in set A.)

3. Out of the three recreations, gardening, reading, and theatre-going, 60 people were asked to say in which they were interested. If T = set of theatre-going people, G = set of gardeners, R = set of readers, and $n(T \cap R) = 16$, $n(R \cap G) = 19$, $n(G \cap T) = 17$, $n(T) = 30$, $n(R) = 43$, $n(G) = 32$, where $n(G)$ denotes the number in set G, etc., find $n(T \cap R \cap G)$. Each of the 60 persons had an interest in at least one of the recreations. (Draw three circles. Use x as in previous examples.)

4. F = set of French students, and $n(F) = 70$; S = set of Science students, and $n(S) = 74$; L = set of Latin students, and $n(L) = 60$; $n(F \cap S) = 29$, $n(S \cap L) = 15$, $n(L \cap F) = 20$, and $n(F \cap S \cap L) = 12$. Find the total number of students under discussion.

5. In a school, 42% of the students take Botany, 40% take Chemistry, and 32% take Zoology; 13% take Botany and Chemistry; 17% take Chemistry and Zoology; and 12% take Zoology and Botany. 23% take none of these subjects.

Use a Venn diagram to find the percentage of students who take:

(*a*) all three subjects;
(*b*) Chemistry only;
(*c*) Botany only;
(*d*) Zoology only;
(*e*) Botany and Chemistry only;
(*f*) Botany and Zoology only;
(*g*) Zoology and Chemistry only.

6. In a survey of students, the percentages studying various languages were found to be as follows: Greek only, 18%; Greek, not Latin, 23%; Greek and French, 8%; Greek, 26%; French, 48%; French and Latin, 8%. 24% of the students studied none of these languages.

Find the percentage studying:

(*a*) Latin only;
(*b*) French only;
(*c*) Greek and Latin;
(*d*) Greek, Latin and French.

7. From your answers to question 6 find the percentage indicated by each of the following sets:

(*a*) $G \cap L \cap F$; (*b*) $G' \cap L \cap F$; (*c*) $G \cap L' \cap F$;
(*d*) $G \cap L \cap F'$; (*e*) $G' \cap L' \cap F$; (*f*) $G' \cap L \cap F'$;
(*g*) $G \cap L' \cap F'$; (*h*) $G' \cap L' \cap F'$.

(NOTE: These classification problems may be solved also by use of the formulae given on page 56.)

CHAPTER VII

Difference of Two Sets

$A-B$ or $A \cap B'$

A not B = $A-B$. If A and B are sets, $A-B$ is called the *set difference* of A and B.

Example 1

If $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and
$B = \{3, 4, 5, 6, 7, 8\}$, then $A-B = \{1, 2, 9, 10\}$

The set difference for these two sets is illustrated by the Venn diagram.

$A-B = \{1, 2, 9, 10\}$

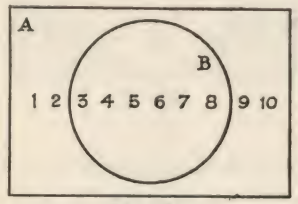


FIG. 24

Example 2

If $C = \{1, 2, 3, 4, 5\}$, and $D = \{3, 4, 5, 6, 7, 8\}$, then

$C-D = \{1, 2\}$

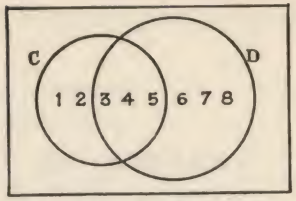


FIG. 25. C−D={1, 2}

(*Caution*: In set notation, C−D does *not* mean that the elements in D are subtracted from those in C. It means the set of the elements in C which are not found in D.)

Example 3

U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10};
A = {4, 5, 6, 7, 8}; and U−A = {1, 2, 3, 9, 10}.

The complement of A (written A′), relative to the given universe, is {1, 2, 3, 9, 10}. Hence U−A = A′.

A′ is the shaded part. U−A is the shaded part (figure 26).

Therefore A′ = U−A.

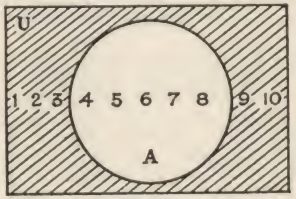


FIG. 26

Definition

A−B = the set of all elements in A which are not in B.

$A-B = \{x \mid x \in A, \text{ and } x \notin B\}$

Example 4

U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
A = {1, 2, 3, 4, }
B = {3, 4, 5, 6, 7}

To illustrate by a Venn diagram that, in the given case, A−B = A ∩ B′.

B′ = {1, 2, 8, 9, 10}
A = {1, 2, 3, 4}
A ∩ B′ = {1, 2}
A−B = {1, 2}
∴ A−B = A ∩ B′

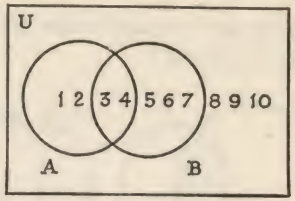


FIG. 27

Note A ∩ B′ = elements in A and B′.

(It must be stressed that the foregoing is not a general proof that A−B = A ∩ B′, but only an illustration for particular sets. A general proof follows.)

General Proof

To prove A−B = A ∩ B′.

(i) $A-B = \{x \mid x \in A \text{ and } x \notin B\}$ by definition of A−B.

(ii) $B' = \{x \mid x \notin B, x \in U\}$.
But $x \notin B$ implies $x \in B'$,
∴ by substituting $x \in B'$ for $x \notin B$ in (i) we get:

(iii) $A-B = \{x \mid x \in A \text{ and } x \in B'\}$.

But by definition

(iv) $A \cap B' = \{x \mid x \in A \text{ and } x \in B'\}$.
∴ A−B = A ∩ B′.

EXERCISE 6

1. If A = {1, 3, 5, 7} and B = {1, 3}, find the set of all elements of A which are not elements of B.

2. Given U = {1, 2, 3, . . . 10}, A = {1, 2, 3, 4}, and B = {3, 4, 5, 6}
write down (i) B′; (ii) A ∩ B′; (iii) A−B.

3. Give a general proof of the statement $A - B = A \cap B'$.

4. What sets are represented by the shaded parts in the following diagrams?

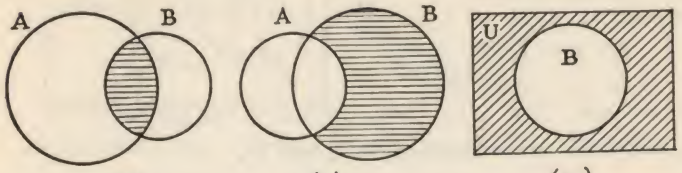


FIG. 28

CHAPTER VIII

Union

∪ For Union

If $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, then putting the two sets together we get a set $C = \{1, 2, 3, 4, 5, 6\}$. The set C, which is called the UNION of sets A and B, is written $A \cup B$, and is read "A *union* B", or "A *cup* B".

(The symbol $\cup$ for union must not be confused with the capital U used for the universe or universal set.)

Again, if $S = \{1, 2, 3, 4, 5\}$ and $R = \{4, 5, 6, 7, 8, 9\}$, the union of S and R is the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

This is illustrated by the Venn diagram (figure 29). The shaded part denotes $S \cup R$.

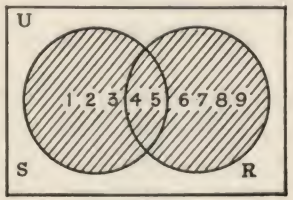


FIG. 29

NOTE: Elements which occur in more than one of the sets—as, in this case, 4 and 5—are listed *only once* in the union of the of the sets.

$$S \cup R = \{x \mid x \in S \text{ or } x \in R \text{ or } x \in \text{both S and R}\}$$

The shaded part of figure 30 represents A ∪ B, that is, every element in A, or in B, or in both A and B; in other words, every element that is in at least one of the sets.

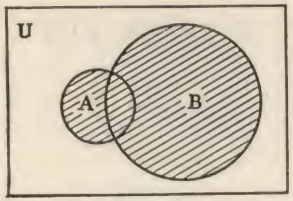


FIG. 30

The unshaded part represents the *complement* of A ∪ B with respect to universe U.

The complement of A ∪ B is written (A ∪ B)′.

Consider A ∪ B, where A and B are disjoint.

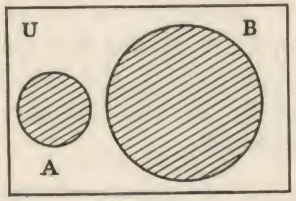


FIG. 31

Suppose A is the set of completely bald men, and B is the set of red-haired men. A ∪ B is the set of men who satisfy one of two conditions: (i) to be completely bald, or (ii) to be red-haired. Since in this case A and B are disjoint, a member of the union cannot satisfy more than one condition. (Note, however, that if A and B were not disjoint—if, for instance, B were the set of men over 50, some members of the union would satisfy *both* conditions.)

The unshaded part of figure 31, called the complement of A ∪ B, and written (A ∪ B)′, is the set of men who fulfil

neither condition; that is, they are not completely bald nor are they red-haired.

If S = the set of people who can sing, and H = the set of people who can play the harp, then S ∪ H is the set of people who can sing, or play the harp, or can both sing and play the harp.

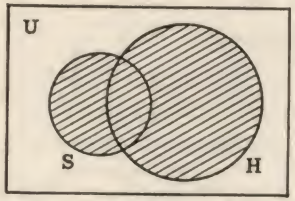


FIG. 32

NOTE: The word "or" has two meanings:

1. The exclusive "or" as in "Tomorrow I shall go to London or to New York" means I shall go to one place or the other, but not to both.
2. The inclusive "or" occurs in the statement: "Students will be selected for the school concert from those who can sing or play the harp." This means *one or other or both.*

   The same idea (one or other or both) is often conveyed by the use of the words "or/and". Mathematicians have agreed, in discussing the union of sets, to interpret "or" as the inclusive "or".

The shaded part of figure 33 shows the union F ∪ E of two sets, F and E, where F is a subset of E.

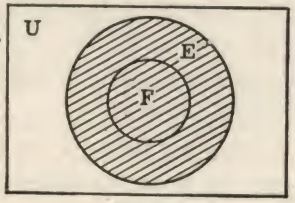


FIG. 33

Suppose F = set of French people, E = set of all Europeans, and the Universe is the set of all the people in the world.

$F \cup E$ is the set of all elements which belong to F or to E or to both. Here $F \cup E = E$.

$(F \cup E)' = E'$ = set of non-Europeans (the unshaded part of the universal set U).

Complement of Intersection

Remember that $\cap$ denotes intersection.

Look at the two sets represented below:

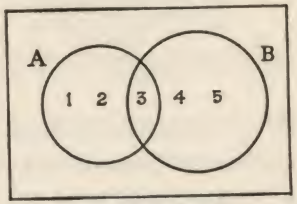


FIG. 34

SET	COMPLEMENT
$A = \{1, 2, 3\}$ $B = \{3, 4, 5\}$ $A \cap B = \{3\}$	$A' = \{4, 5\}$ $B' = \{1, 2\}$ $(A \cap B)' = \{1, 2, 4, 5\}$ or The complement of the intersection $= \{1, 2, 4, 5\}$

But the union of A′ and B′ is also $\{1, 2, 4, 5\}$.

Therefore $(A \cap B)' = A' \cup B'$.

The complement of the intersection of A and B equals the union of the complements of A and B.

NOTE: Remember that the above is not a general proof that $(A \cap B)' = A' \cup B'$. It is an illustration in a special case where $A \cup B = U$.

Illustration of

$(A \cap B)' = A' \cup B'$

Suppose A = the set of all thin people, and B = set of all tall people.

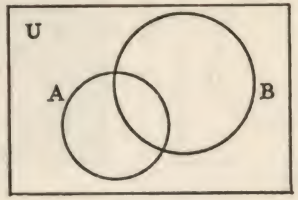
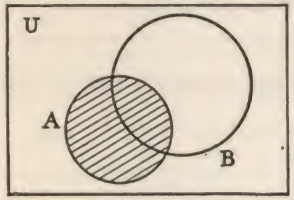
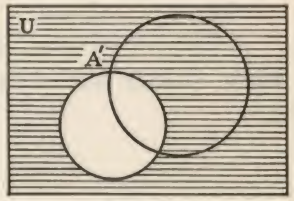
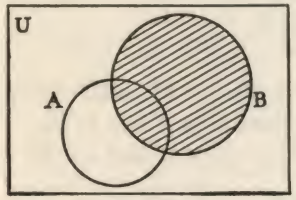
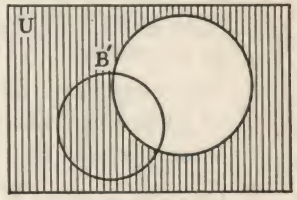
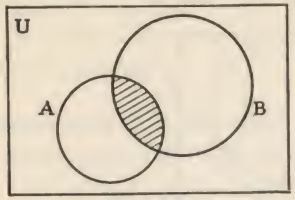
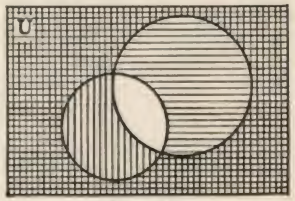


FIG. 35a.

SET	COMPLEMENT OF SET
 FIG. 35b. A = thin people	 FIG. 35e. A′ = people who are not thin. (indicated by shading)
 FIG. 35c. B = tall people	 FIG. 35f. B′ = people who are not tall

SET	COMPLEMENT OF SET
 FIG. 35d.	 FIG. 35g.
$A \cap B$ = people who are both thin and tall. The unshaded part represents $(A \cap B)'$	$(A \cap B)'$ = people who are not tall and not thin = A' superimposed on B' and is therefore the union of A' and $B' = A' \cup B'$

NOTE: Shaded part of figure 35g = unshaded part of figure 35d.

SUMMARY

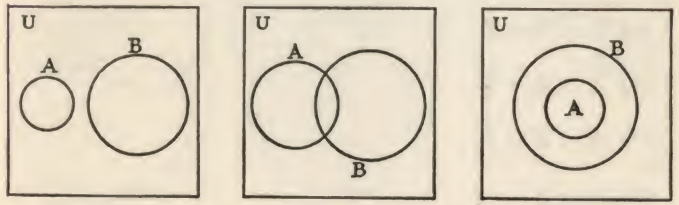
1. $\cup$ denotes the union of sets.
2. $A \cup B$ is the set of all elements which belong to A, or to B, or to both A and B.
   In set notation: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
3. The complement of $A \cup B$, written $(A \cup B)'$ is the set of all elements which are neither in A nor in B.
   In set notation $(A \cup B)' = \{x \mid x \notin A \text{ and } x \notin B\}$
4. $(A \cap B)' = A' \cup B'$ means
   "The complement of the intersection of A and B equals the union of the complements of A and B."

EXERCISE 7

1. In the universe $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
$A = \{1, 2, 3, 4\}$
and $B = \{3, 4, 5, 6\}$

(*a*) Find $A \cap B$.
(*b*) Find $B \cap A$.
(*c*) Is $A \cap B = B \cap A$?
(*d*) Find $A \cup B$.
(*e*) Find $B \cup A$.
(*f*) Is $A \cup B = B \cup A$?
(*g*) List the elements in A'.
(*h*) List the elements in B'.
(*i*) List the elements in $A' \cup B'$.
(*j*) List the elements in $(A \cap B)'$.
(*k*) Is $A' \cup B' = (A \cap B)'$?
(*l*) Show that $A \cup A' = U$.
(*m*) Show that $A \cap A' = \emptyset$.

2. Make copies of the following Venn diagrams and shade the parts that indicate the union of A and B, that is, $A \cup B$.



(i) (ii) (iii)

FIG. 36

3. Make copies of the diagrams in question 2, and use shading to indicate the intersection of A and B, that is $A \cap B$.

4. Make a copy of figure 37 and use shading to show that $A \cup U = U$.

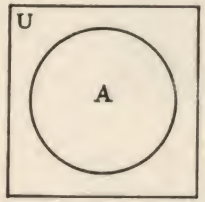


FIG. 37

5. Make another copy of figure 37 and use shading to indicate A'.

6. With a similar diagram, use shading to show that

$$A \cup A' = U$$

7. Draw pairs of circles to represent two sets A and B so situated that the following results follow:

(*a*) $A \cap B = \emptyset$

(*b*) $A \cap B = B$

(*c*) $A \cap B = A$

(*d*) $A \cup B = A$

(*e*) $A \cup B = B$

8. Identify at sight in terms of set operations the shaded part of each of the following diagrams:

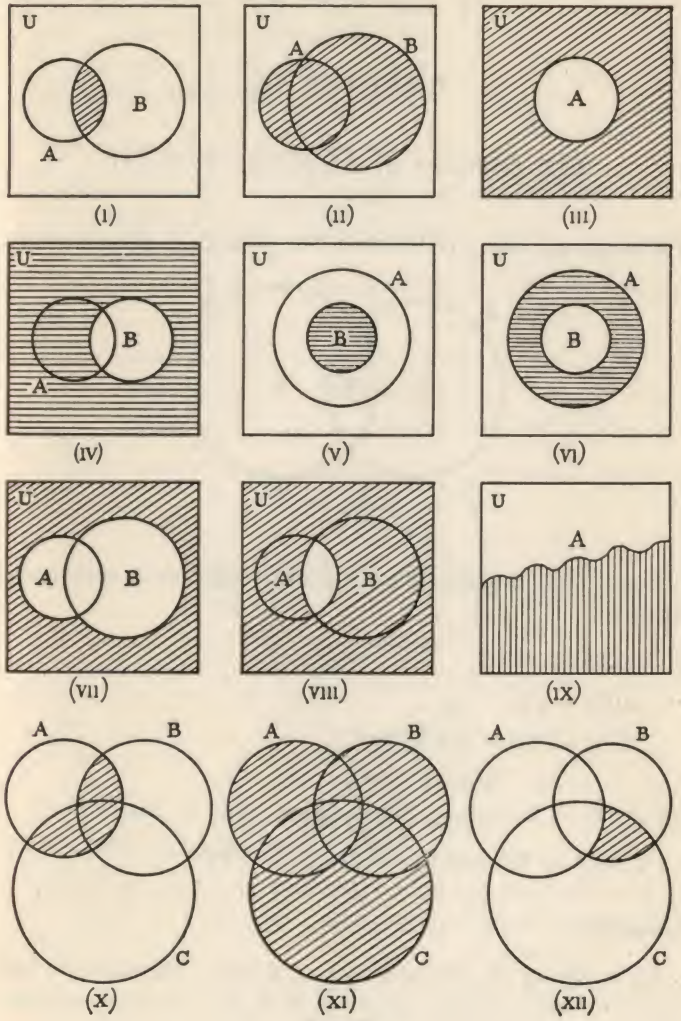


FIG. 38

CHAPTER IX

The Number of Elements in a Set

If A is any set, then $n(A)$ denotes the number of elements in A.

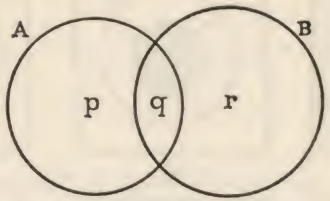


FIG. 39

If p, q, and r represent the number of elements in each space, we have:

$$n(A) = p+q$$
$$n(B) = q+r$$
$$\begin{aligned} n(A)+n(B) &= p+2q+r \\ &= p+q+r+q \\ &= n(A \cup B)+n(A \cap B) \end{aligned}$$
$$\therefore n(A \cup B) = n(A)+n(B)-n(A \cap B)$$

Examples

In a school of 300 students, 170 take Science, 230 take Classics. Each student takes at least one of these subjects. How many take both?

Let S = set of Science students.
C = set of Classics students.

$$n(S \cup C) = n(S)+n(C)-n(S \cap C)$$
$$300 = 170+230-n(S \cap C)$$
$$-100 = -n(S \cap C)$$

Therefore 100 students take both.

For any three sets A, B, and C:

$$n(A \cup B \cup C) = n(A)+n(B)+n(C)-n(A \cap B) -n(B \cap C)-n(C \cap A)+n(A \cap B \cap C)$$

The truth of this can be illustrated by diagram for a particular case.

$$\begin{array}{lllllllll} n(A) = 1+ & \boxed{2} & & & + & \boxed{5} & + & \boxed{6} & \\ n(B) = & \boxed{2} & +3+ & \boxed{4} & + & \boxed{5} & & & \\ n(C) = & & & \boxed{4} & + & \boxed{5} & + & \boxed{6} & +7 \end{array}$$

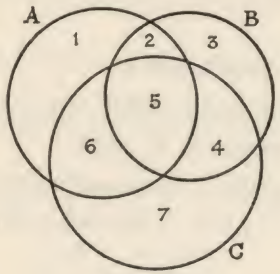


FIG. 40

In the sum of $n(A)+n(B)+n(C)$, 2, 4 and 6 are counted twice, and 5 is counted three times.

$$\begin{aligned} n(A \cup B \cup C) &= n(A)+n(B)+n(C)-(2+4+6+5+5) \\ &= n(A)+n(B)+n(C)-(2+5)-(4+5) \\ &\qquad -(6+5)+5 \\ &= n(A)+n(B)+n(C)-n(A \cap B)-n(B \cap C) \\ &\quad -n(C \cap A)+n(A \cap B \cap C) \end{aligned}$$

Example

A school record shows that 30% of the students take German, 29% take French, and 31% Spanish; 8% take German and French, 11% take French and Spanish, 9% take Spanish and German, and 3% take all three languages. Assuming that each student takes at least one of these languages, find if these figures are consistent.

$$\begin{aligned} n(G \cup F \cup S) &= n(G)+n(F)+n(S)-n(G \cap F)-n(F \cap S) \\ &\quad -n(S \cap G)+n(G \cap F \cap S) \\ &= 30+29+31-8-11-9+3 = 65 \end{aligned}$$

Result: The figures are *not* consistent, since they show that 35% of the students do not take any of the three languages.

The two formulae given below may be used as an alternative method of solving the counting problems in Exercise 5, pp. 39–40.

SUMMARY

For any two sets A and B:

$$n(A \cup B) = n(A)+n(B)-n(A \cap B)$$

For any three sets A, B, and C:

$$n(A \cup B \cup C) = n(A)+n(B)+n(C)-n(A \cap B) -n(B \cap C)-n(C \cap A)+n(A \cap B \cap C).$$

CHAPTER X

The Real Numbers

A rational number is one which can be expressed as a *ratio*. This is why it is called "rational".

A fraction is a ratio. The fraction $\frac{1}{2}$, for example, expresses the ratio 1 : 2 and is a rational number.

Any number which can be written in the form $\frac{a}{b}$, that is, as a ratio, where a and b are integers, and where b is not zero, is a rational number.

Thus ·75 is a rational number, since it can be written as $\frac{75}{100}$ (or as $\frac{3}{4}$, $\frac{6}{8}$, etc.).

$2\frac{1}{3}$ is a rational number, since it can be written as $\frac{7}{3}$ (or as $\frac{14}{6}$, $\frac{28}{12}$, etc.).

9 is a rational number, since it can be written as $\frac{9}{1}$ (or as $\frac{18}{2}$, $\frac{36}{4}$, etc.).

But $\sqrt{2}$ is an *irrational* number. It cannot be expressed as a ratio.

$$\begin{aligned} (1{\cdot}4)^2 &= 1{\cdot}96 \\ (1{\cdot}41)^2 &= 1{\cdot}9881 \\ (1{\cdot}414)^2 &= 1{\cdot}999396 \end{aligned}$$

$\frac{14}{10}$ is an approximate value for $\sqrt{2}$; $\frac{141}{100}$ is a closer approximation; $\frac{1414}{1000}$ is a very close approximation. We can get still closer approximations, but there is no fraction (or ratio) which is equal to $\sqrt{2}$. In other words $\sqrt{2}$ cannot be written as a fraction, and so is *irrational*. $\frac{\sqrt{2}}{3}$, and $\frac{\sqrt{3}}{4}$ are also irrational.

Real numbers include the rational and irrational numbers.

The Real Number Line

Consider the line in the drawing below as extending endlessly in both directions as indicated by the arrows.

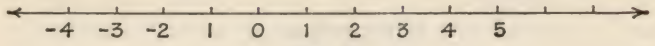


FIG. 41

We select any point and label it 0. We select a point to the right of 0 and label it 1. The length of the line from 0 to 1 we call the unit length. This unit length is marked off repeatedly to the right giving points corresponding to the *counting numbers* (sometimes called the natural numbers) 1, 2, 3, . . .

Similarly, starting again at 0, we use the unit length to mark off $-1, -2, -3, \ldots$

The infinite set of whole numbers or integers to the right of 0 may be written in set notation as follows:

$S = \{1, 2, 3, 4, \ldots\}$ or $S = \{x \mid x \text{ an integer}, x > 0\}$.

Similarly, the infinite set of integers to the left of 0 may be written in set notation:

$S = \{-1, -2, -3, \ldots\}$ or $S = \{x \mid x \text{ an integer}, x < 0\}$.

The complete set of these numbers on the real number line, consisting of negative integers, zero, and positive whole numbers, is called the *set of integers.*

We can write the set of integers as

$I = \{\ldots . -4, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

or $I = \{x \mid x \text{ an integer}, x < = \text{ or } > 0\}$.

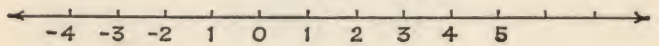


FIG. 42

Rational Numbers

Returning again to the real number line we can subdivide each unit starting at 0 into the fractions $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ Thus:

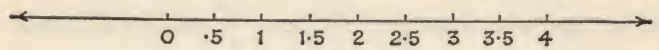


FIG. 43

We can continue this process for fractions with denominators of 3, 4, 5, 6, ...

Thus we can associate every rational number, positive or negative, with a point on the number line; in other words, we can establish a one-to-one correspondence between the rational numbers and some of the points on the line.

But, while we can establish a one-to-one correspondence between every rational number and a point on the real number line, we *cannot* establish a one-to-one correspondence between every point and a rational number.

No matter how far we continue our subdivision with fractions having enormously great denominators, we cannot establish a one-to-one correspondence with real numbers like $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \ldots$ because such numbers, being irrational, cannot be written in the form of fractions or ratios.

This idea, that there are very many gaps in the number line which we *cannot* label as representing fractions, is most important in Infinite Set Theory.

Graphical Representation of $\sqrt{2}$

Although $\sqrt{2}$ cannot be written as a rational number or as a decimal, it may be represented on the number line.

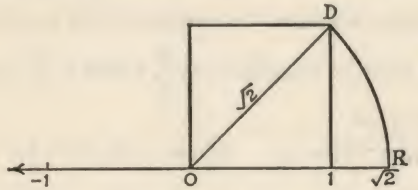


FIG. 44

Construct a square on the unit line between O and 1. Join the diagonal OD. With centre O and radius OD describe an arc cutting the number line at R. The point R represents the irrational number $\sqrt{2}$. The point R on the number line has no *rational* number corresponding to it.

Proof that $\sqrt{2}$ is Irrational

Suppose that $\sqrt{2} = \frac{p}{q}$ where p and q are integers and contain no common factor.

Squaring both sides, we get $2 = \frac{p^2}{q^2}$ or $2q^2 = p^2$.

This means that p^2 is an even number. Then p must be an even number because no two odd numbers multiplied together can given an even result.

Substituting $2r$ for p (r being an integer), we get

$$2q^2 = 4r^2 \quad \text{or} \quad q^2 = 2r^2$$

This means that q must be an even number.

Therefore we supposed an impossibility when we supposed that $\sqrt{2} = \frac{p}{q}$ where p and q contain no common factor.

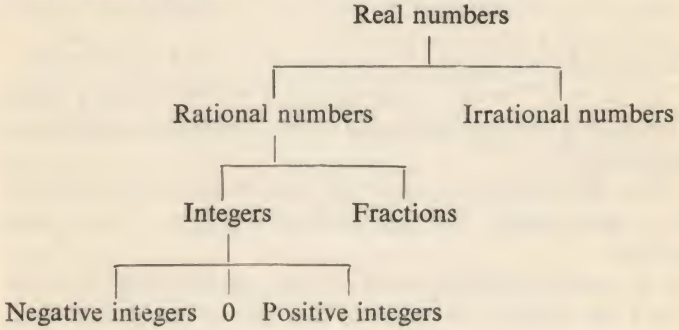
Therefore $\sqrt{2} \neq \frac{p}{q}$

SUMMARY

1. A rational number is a number which can be expressed in the form of a fraction, as $\frac{a}{b}$, where $b \neq 0$ and a and b are integers.
   Examples: $7\ (=\frac{7}{1})$; $-\cdot75\ (=-\frac{3}{4})$; $4\ (=\frac{8}{2})$; $\dot{3}\ (=\frac{1}{3})$.
2. An irrational number is a number which CANNOT be expressed in the form of a fraction $\frac{a}{b}$ where $b \neq 0$ and a and b are integers.
   Examples: $\sqrt{2}$; $\frac{1}{\sqrt{3}}$; $\frac{\sqrt{7}}{3}$.

3. Every rational number can be associated with a point on the number line, but every point on the number line cannot be associated with a rational number.
   For example, the point R in figure 44 cannot be associated with a rational number.

The Number System



CHAPTER XI

New Names for Old

"The sum of 7 and 10 is 17" is a sentence. It is a *true* sentence.

"$x+10 = 17$" is also a sentence; it is neither true nor false. It becomes a true sentence only when x (called a *variable*) is replaced by 7.

$\{7\}$ is the *replacement set*, or the *solution set*, or the *truth set*, for x, that changes the *open sentence* $x+10 = 17$ to a true sentence.

If a number sentence involves any symbol such as x or y, and the symbol can refer to any one of many numbers, the sentence is called an *open sentence*. It is not necessarily true or false; it leaves the matter open for further consideration.

$x^2-25 = 0$ is an open sentence. When x is replaced by either 5 or -5, the sentence is true.

$\{5, -5\}$ is called the replacement set of the equation $x^2-25 = 0$.

The traditional language of algebra has undergone a change in order to bring it more into line with Set Theory. The following parallel columns show some of the changes.

OLD	NEW
1. An equation	1. An open sentence
2. Solve the equation $x+4 = 9$ Answer $x = 5$. 5 is called the root of the equation $x+4 = 9$.	2. Find the replacement set (or the solution set, or the truth set) of the open sentence $x+4 = 9$. Answer $\{5\}$. $\{5\}$ is called the replacement set, or the solution set, of the open sentence $\{x \mid x+4 = 9\}$.
3. Solve the equation $x^2-7x+12 = 0$ $(x-3)(x-4) = 0$ $\therefore\ x = 3$ or 4 3 and 4 are called the roots.	3. Find the solution set of $\{x \mid x^2-7x+12 = 0\}$ $(x-3)(x-4) = 0$ $\therefore\ x = 3$ or 4 $\{3, 4\}$ is the replacement set (or the solution set).
4. Solve the equation $7x+1 = 13$ Ans. $x = \frac{1}{7}\frac{2}{}$	4. Find the solution set of $\{x \mid 7x+1 = 13, x \text{ real}\}$ Ans. $\{\frac{1}{7}\frac{2}{}\}$

NOTE: If, in No. 4 at the right, the problem had been: Find the solution set of $\{x \mid 7x+1 = 13, x \text{ an integer}\}$, then the answer would have been $\emptyset$ because there is no *integer* that can replace the x in $7x+1 = 13$ so as to make $7x+1 = 13$.

EXERCISE 8

Find the replacement set of each of the following equations or open sentences:

(*a*) $\{x \mid x-5 = 7, x \text{ an integer}\}$
(*b*) $\{x \mid x^2-9 = 0, x \text{ an integer}\}$
(*c*) $\{x \mid x^2-x-20 = 0, x \text{ an integer}\}$
(*d*) $\{x \mid 6x^2-17x+12 = 0, x \text{ an integer}\}$
(*e*) $\{x \mid 6x^2-17x+12 = 0, x \text{ a rational number}\}$
(*f*) $\{x \mid 2x^2+3x+5 = 0, x \text{ real}\}$
(*g*) $\{x \mid 2x^2+3x-5 = 0, x \text{ rational}\}$
(*h*) $\{x \mid 5x^2-6x-7 = 0, x \text{ real}\}$
(*i*) $\{x \mid 5x^2-6x-7 = 0, x \text{ rational}\}$

CHAPTER XII

Graphs of Inequalities (i)

1. Draw the graph of $x > 2$ with universe $U = \{1, 2, 3, 4, 5\}$.

Here we are concerned only with the numbers in the universal set, namely 1, 2, 3, 4, and 5.

Figure 45 is the required graph:

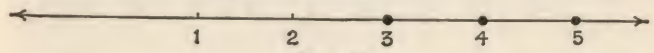


FIG. 45

The heavy dots on the number line indicate that the solution set within the given universe U is $\{3, 4, 5\}$, that is, three isolated points.

2. Draw the graph $x > 2$ where the universe is the set of real numbers.

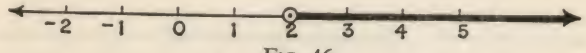


FIG. 46

(*a*) The open circle at 2 tells us that 2 is not a member of the solution set.

(*b*) The heavy black line with the arrow indicates that every point to the right of 2 is a member of the solution set. There are no isolated points.

3. Represent graphically $x \geqq 2$ with U = real numbers.

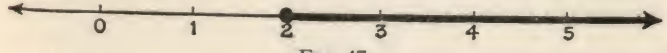


FIG. 47

(*a*) The solid black circle indicates that 2 is included in the solution set.

(*b*) The heavy black line with the arrow indicates that every point to the right of 2 is a member of the solution set. There are no isolated points.

Oral Exercise

What is the solution set in each of the following graphs?

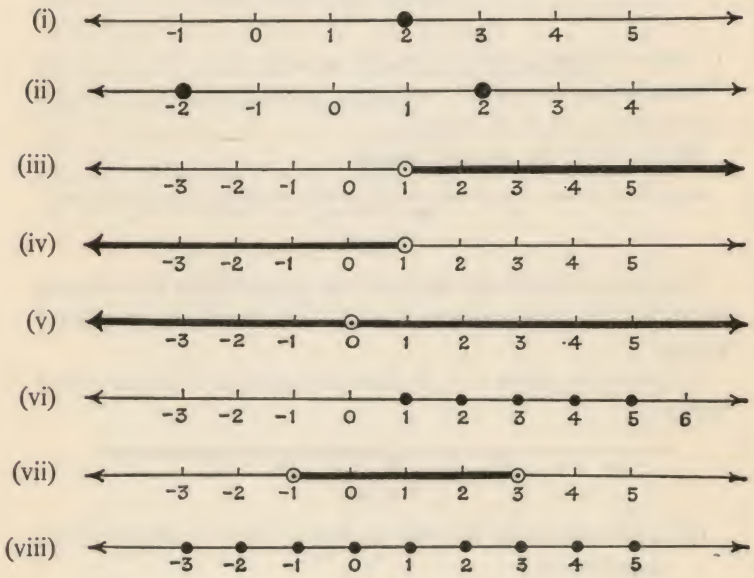


FIG. 48

EXERCISE 9

Graph the following:

1. $\{x \mid x > 4\}$, U = $\{2, 3, 4, 5, 6\}$
2. $\{x \mid x > 3\}$, U = real numbers
3. $\{x \mid x \geqq 4\}$, U = $\{2, 3, 4, 5, 6\}$

4. $\{x \mid x \geqq 3\}$, U = real numbers
5. $\{x \mid 3x-2 = x+2\}$, U = real numbers
6. $\{x \mid 3x-2 > 7\}$, U = $\{2, 3, 4, 5\}$
7. $\{x \mid x^2-9 = 0\}$, U = $\{-4, -3, -2 \ldots +4\}$
8. $\{x \mid x^2-9 = 0\}$, U = real numbers
9. $\{x \mid 3x-2 > 13\}$, U = real numbers
10. $\{x \mid 3x-2 < 13\}$, U = real numbers

CHAPTER XIII

Operations with Inequalities

We use each of the symbols $>$ and $<$ so that the pointed end is directed towards the smaller number, and the wide or open end towards the larger number.

$9 > 7$ is the same statement as $7 < 9$.
$x > 0$ denotes that x is positive.
$x < 0$ denotes that x is negative.

I. $17 > 13$

(i) Add 10 to each side

$$17+10 > 13+10$$

(ii) Subtract 10 from each side

$$17-10 > 13-10$$

Adding the same number to each side of an inequality, or subtracting the same number from each side, does not alter the inequality.

II. $5 > 3$

(i) Multiply by 10

$$50 > 30$$

(ii) Divide by 3

$$\frac{5}{3} > 1$$

Multiplying or dividing each member of an inequality by the same *positive* number does not alter the inequality.

BUT

III. $5 > 3$

(i) Multiply by -10

$$-50 < -30$$

(ii) Divide by -3

$$-\frac{5}{3} < -1$$

Multiplying or dividing each side of an inequality by the same *negative* number alters the inequality, that is, $>$ becomes $<$, and *vice versa.*

Examples

(*a*) $7x-5 > 9$
$7x > 9+5$ . . . add $+5$ to each side,
$7x > 14 \quad \therefore \; x > 2$

(*b*) $3x+2 < 11$
$3x < 11-2$ . . . subtract 2 from each side,
$3x < 9 \quad \therefore \; x < 3$

(*c*) $-x > 4$
$x < -4$ . . . Multiply each side by -1 and change the inequality.

If both members of an inequality are multiplied or divided by the same negative number, the direction of the inequality must be reversed, or $>$ changed to $<$ and $<$ changed to $>$

This statement has not been proved; it has been illustrated for particular numbers.

Examples

Consider the solution sets and graphs of the following:

(*a*) $\{x \mid (x-1)(x-4) = 0\}$

(*b*) $\{x \mid (x-1)(x-4) > 0\}$

(*c*) $\{x \mid (x-1)(x-4) < 0\}$

(*a*) Solution set is $\{1, 4\}$, and the graph shows 1 and 4 as two black dots. The solution set is two isolated points.

(*b*) $(x-1)(x-4) > 0$. [Regard "> 0" as "is positive".] For $(x-1)(x-4)$ to be positive these two factors must be (i) both positive $(+)(+)$, or (ii) both negative $(-)(-)$.

If $x-1 > 0$, then $x > 1$ and if $x-4 > 0$, then $x > 4$ } Both statements included in $x > 4$

If $x-1 < 0$, then $x < 1$ and if $x-4 < 0$, then $x < 4$ } Both statements included in $x < 1$

Solution set of $\{x \mid x > 4 \quad \text{or} \quad x < 1, x \text{ real}\}$.

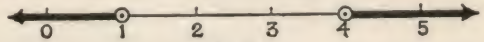


FIG. 49

(i) The open circles at 1 and 4 mean that 1 and 4 are *not* members of the solution set.

(ii) The heavy black lines and their arrows indicate that all real numbers less than 1 and greater than 4 are elements of the solution set.

(*c*) $(x-1)(x-4) < 0$. [Regard "< 0" as "is negative".] For $(x-1)(x-4)$ to be negative, x must lie between 1 and 4, that is $1 < x < 4$.

Solution set of $\{x \mid 1 < x < 4, x \text{ real}\}$.

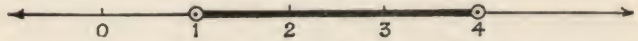


FIG. 50

(i) The open circles at 1 and 4 indicate that 1 and 4 are not members of the solution set.

(ii) The heavy line segment from 1 to 4 indicates that all real numbers greater than 1 and less than 4 are members of the solution set.

EXERCISE 10

1. Find the solution set of each of the following, and illustrate each answer by a graph on the number line:
   (i) $A = \{x \mid x > 5, x \text{ an integer}\}$, $U = \{4, 5, 6 \ldots 9\}$
   (ii) $B = \{x \mid x \geqq 5, x \text{ an integer}\}$, $U = \{4, 5, 6 \ldots 9\}$
   (iii) $C = \{x \mid x < 5, x \text{ an integer}\}$, $U = \{0, 1, 2, 3, 4\}$
   (iv) $D = \{x \mid x > 5, x \text{ a real number}\}$
   (v) $E = \{x \mid x \geqq 5, x \text{ a real number}\}$
2. A is the set $\{x \mid x < 9\}$, B is the set $\{x \mid x-3 > 0\}$. The universe is R (the real numbers). Indicate each of these sets on the same number line.
3. From your answer to question 2 find $A \cap B$ and $A \cup B$.
4. Show by a graph on the number line that the graph of the set $\{x \mid 3 \leqq x \leqq 7\}$ is the intersection of the sets
   $$\{x \mid 3 \leqq x\} \quad \text{and} \quad \{x \leqq 7\}$$
5. Mark on the number line the solution set of the equality $(x-4)(x+3) = 0$.
6. Indicate on the number line the solution set of the inequality $(x-4)(x+3) > 0$ for all real values of x.
7. From your graph in question 6 show that
   $$\{x \mid x-4 > 0\} \cup \{x \mid x+3 < 0\}$$
   is the solution set of
   $$\{x \mid (x-4)(x+3) > 0\}$$

8. From your graph in questions 5 and 6 show that

$$\{x \mid x-4 < 0\} \cap \{x \mid x+3 > 0\}$$

is the solution set of

$$\{x \mid (x-4)(x+3) < 0\}$$

9. By drawing the graph of the solution set for $x > \pm 4$, x real, show that the following statement is false: If $x^2 > 16$, then $x > \pm 4$.

10. Show that the solution set for $x^2 > 16$ is:

$$\{x \mid x > 4 \quad \text{or} \quad x < -4, x \text{ real}\}$$

Find the solution sets of the following equalities and inequalities, and show each solution set by a graph on the number line. (Assume that x is a real number.)

11. $\{x \mid x^2-x-12 = 0\}$

12. $\{x \mid x^2-x-12 > 0\}$

13. $\{x \mid x^2-x-12 < 0\}$

14. $\{x \mid 2x^2+x-15 = 0\}$

15. $\{x \mid 2x^2+x-15 < 0\}$

16. $\{x \mid 2x^2+x-15 > 0\}$

17. $\{x \mid 2x^2+3x+4 = 0\}$

18. $\{x \mid 2x^2+3x+4 > 0\}$

19. $\{x \mid 6x^2-x-12 = 0\}$

20. $\{x \mid 6x^2-x-12 \leqq 0\}$

CHAPTER XIV

The Commutative, Associative, and Distributive Laws for Intersection and Union

Addition, subtraction, multiplication, and division are called *operations* in mathematics. Three fundamental laws hold for the operations addition and multiplication, namely:

The *Commutative Law:* $a+b = b+a$ (for addition);
and $ab = ba$ (for multiplication).

The *Associative Law:* $a+(b+c) = (a+b)+c$ (addition);
and $a(bc) = (ab)c$ (for multiplication).

The *Distributive Law:* $a(b+c) = ab+ac$.

Similar laws hold for corresponding operations with sets.

It is necessary to test whether these three fundamental laws are true in set operations, because in certain branches of mathematics these laws do not apply.

(i) The Commutative Law holds for Union

$$A \cup B = B \cup A = \{x \mid x \in A \text{ or } x \in B \textit{ or } x \in (A \text{ and } B)\}$$

$$A \cup B = B \cup A$$

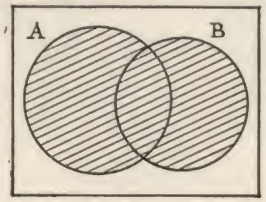


FIG. 51

(ii) The Commutative Law holds for Intersection

$$A \cap B = B \cap A = \{x \mid x \in A \text{ and } x \in B\}$$

$$A \cap B = B \cap A$$

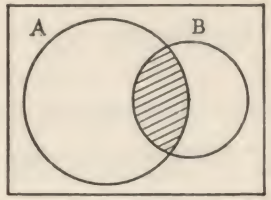


FIG. 52

(iii) The Associative Law for Three Sets for Union

To show by Venn diagrams that

$$A \cup (B \cup C) = (A \cup B) \cup C$$

The bracketed operations are represented first.

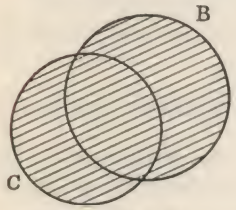


FIG. 53 $B \cup C$

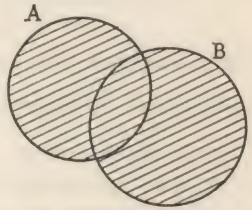


FIG. 54 $A \cup B$

Then we complete the diagrams, getting two identical figures.

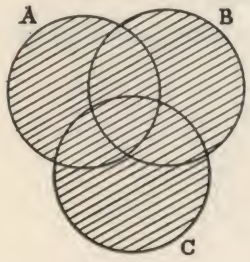


FIG. 55

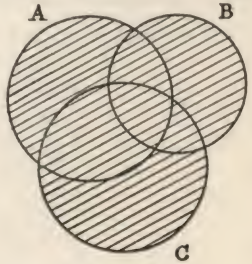


FIG. 56

$$A \cup (B \cup C) = (A \cup B) \cup C$$

(iv) The Associative Law for Three Sets for Intersection

To show by a Venn diagram that

$$A \cap (B \cap C) = (A \cap B) \cap C$$

The bracketed operations are represented first.

Indicate $B \cap C$ by horizontal shading, as in figure 57. Then indicate $A \cap (B \cap C)$ by vertical shading.

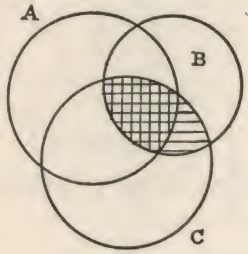


FIG. 57

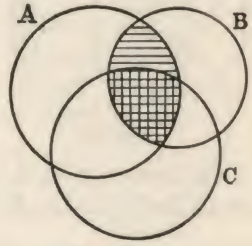


FIG. 58

Indicate $A \cap B$ by horizontal shading, as in figure 58. Then indicate $(A \cap B) \cap C$ by vertical shading.

The double-hatched portions in figures 57 and 58 are identical.

$$A \cap (B \cap C) = (A \cap B) \cap C$$

SUMMARY

Union is a commutative operation:

$$A \cup B = B \cup A$$

Intersection is a commutative operation:

$$A \cap B = B \cap A$$

Union is an associative operation:

$$A \cup (B \cup C) = (A \cup B) \cup C$$

Intersection is an associative operation:

$$A \cap (B \cap C) = (A \cap B) \cap C$$

EXERCISE 11

1. Using the method of double shading as in the previous examples illustrate that $A \cap (B \cap C) = (A \cap B) \cap C$ for the three sets A, B, C shown in figure 59.

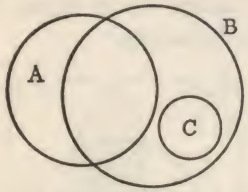


FIG. 59

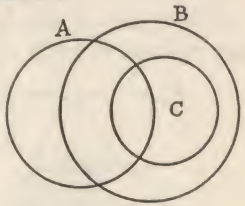


FIG. 60

2. For the three sets A, B, and C shown in figure 60, show that $A \cap (B \cap C) = (A \cap B) \cap C = \emptyset$

Multiplication is Distributive over Addition.
Intersection is Distributive over Union.

Compare these parallel columns:

Multiplication distributes over addition	**Intersection distributes over union**
Numerical algebra with *a*, *b*, and *c* denoting numbers.	Set algebra with A, B, and C denoting sets.
$a(b+c) = ab+ac$. This is familiar to the student. The multiplier *a* outside the bracket is distributed over the *b* and *c* inside the bracket giving $ab+ac$. This is what is meant when we say that in arithmetical algebra "Multiplication is distributive over addition" or "Multiplication is distributive with respect to addition".	$A \cap (B \cup C) =$ $(A \cap B) \cup (A \cap C)$ The A $\cap$ outside the bracket is distributed over the B and C inside giving $(A \cap B) \cup (A \cap C)$ This is what is meant when we say that in the algebra of sets "Intersection is distributive over union" or "Intersection is distributive with respect to union".

Remembering the formula $a(b+c) = ab+ac$ will help us to remember that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

To show by means of Venn diagrams that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

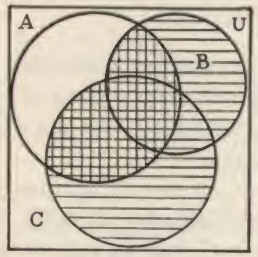


FIG. 61

$B \cup C$ is represented by horizontal shading. The double-hatched portion is $A \cap (B \cup C)$.

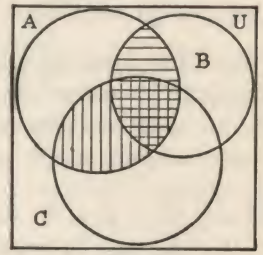


FIG. 62

$A \cap B$ is represented by horizontal shading, and $A \cap C$ by vertical shading. The complete shaded portion is the union of $A \cap B$ and $A \cap C$.

The double-hatched portion in figure 61 is identical with the whole shaded portion in figure 62.

$$\therefore \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

REMEMBER

The union of $(A \cap B)$ and $(A \cap C)$ includes all elements in one or the other or in both.

This concludes the proof, by Venn diagrams, that

INTERSECTION IS DISTRIBUTIVE OVER UNION

EXERCISE 12

Illustrate by a Venn diagram of three overlapping circles A, B, and C, that union is distributive over intersection, that is that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

NOTE: The associative law in set operations states that we can omit the brackets when working *only* with unions of sets, and also when working *only* with intersection, but not when we are combining union and intersection.

Thus: $(A \cup B) \cup C$ may be written $A \cup B \cup C$, and $A \cap (B \cap C)$ may be written $A \cap B \cap C$, but we cannot omit the brackets in $A \cup (B \cap C)$, or in $A \cap (B \cup C)$.

EXERCISE 13

1. U, the set of all the students in a school, has the following subsets:

A = {first-year students}; B = {second-year students};

C = {third-year students}; D = {fourth-year students};

M = {students in the school opera}; T = {students who play tennis}.

Interpret the following subsets:

(*a*) $A \cup B \cup C$; (*b*) $A \cup B \cup C \cup D$;
(*c*) $M \cap (C \cup D)$; (*d*) $(M \cap C) \cup D$; (*e*) $T \cap M$;
(*f*) $M \cap T$; (*g*) $T \cap M'$; (*h*) $T' \cap M'$;
(*i*) $(T \cup M)'$; (*j*) $A \cap M$; (*k*) $A \cap M'$;
(*l*) $(A \cap M) \cup (A \cap M')$

2. If U = {Joe, John, Tom, Pat, Leo, Frank, Len, Dick},
A = {Joe, John, Tom, Pat}, and
B = {Leo, Frank, Len, Dick}, describe:
(*a*) $A \cup B$; (*b*) $A \cap B$; (*c*) A'; (*d*) B'; (*e*) $A' \cup B$;
(*f*) $A \cup B'$; (*g*) $(A \cup B)'$; (*h*) $A' \cap B'$.

3. U = {1, 2, 3, 4, 5, 6, 7}, A = {1, 2, 3, 4},
B = {3, 4, 5, 6}.
Find (*a*) A'; (*b*) B'; (*c*) $A \cap B$; (*d*) $A \cup B$; (*e*) $A' \cup B$;
(*f*) $A \cup B'$; (*g*) $A' \cap B$; (*h*) $A \cap B'$; (*i*) $A' \cup B'$;
(*j*) $(A \cup B)'$; (*k*) $A' \cap B'$.

4. Given that A and B are subsets of universe U, complete the following equations:

(*a*) $A \cup A =$; (*b*) $A \cap A =$; (*c*) $A \cap A' =$;
(*d*) $A \cup A' =$; (*e*) $(A')' =$; (*f*) $((A')')' =$;
(*g*) $A \cup \emptyset =$; (*h*) $A \cap U =$; (*i*) $U' =$;
(*j*) $\emptyset' =$; (*k*) $(A \cup B)' =$; (*l*) $(A \cap B)' =$.

5. Make copies of this Venn diagram and shade them to show the following operations. (Make a separate drawing for

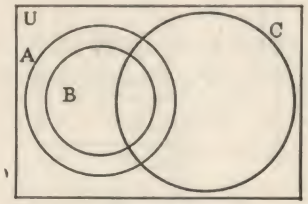


FIG. 63

each operation and write the notation for the operation under your figure.)

(*a*) $A \cap B$; (*b*) $B \cap C$; (*c*) $A \cup B$;
(*d*) $B \cup C$; (*e*) $A \cup C$; (*f*) $(A \cap B)'$;
(*g*) $(B \cap C)'$; (*h*) $(A \cup B)'$; (*i*) $(B \cup C)'$.

6. Using separate Venn diagrams showing three intersecting circles A, B, and C, indicate by shading:

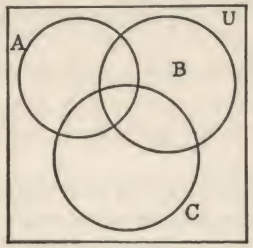


FIG. 64

(*a*) $A \cap (B \cap C)$; (*b*) $(A \cap B) \cap C$;
(*c*) $(A \cup B) \cup C$; (*d*) $A \cup (B \cap C)$;
(*e*) $(A \cup B) \cap C$; (*f*) $A \cap (B \cup C)$;
(*g*) $(A \cup B)'$.

7. Using separate Venn diagrams of three circles, A, B, and C (figure 65), indicate the following operations:

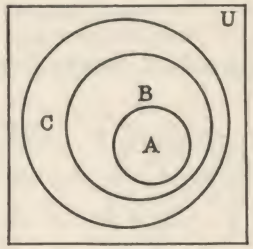


FIG. 65

(*a*) $(B \cap C)'$; (*b*) $A \cup C$;
(*c*) $(A \cap B)'$; (*d*) $A \cap (B \cap C)$;
(*e*) $(A \cap B) \cap C$; (*f*) $(A \cup B) \cup C$;
(*g*) $(A \cup B)'$.

8. Set $A = \{a\}$ has 2 subsets, namely $\emptyset$ and $\{a\}$; set $B = \{a, b\}$ has 2^2 subsets, $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$. Show, by listing them, that set $C = \{a, b, c\}$ has 2^3 or 8 subsets.

Hence show that a set with n elements has 2^n subsets.
[Hint: set B has twice as many subsets as set A; set C has twice as many as set B; etc.]

9. U = set of all plane triangles,
I = set of isosceles triangles,
E = set of equilateral triangles,
R = set of right-angled triangles, and
S = set of scalene triangles.

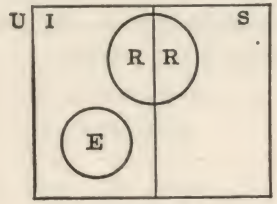


FIG. 66

Write in set notation:

(*a*) the set of isosceles right-angled triangles; (*b*) the set of scalene right-angled triangles; (*c*) the set of equilateral right-angled triangles.

10. With regard to the diagram in figure 66, which of the following statements are true, and which false?

(i) $I \cap E = E$; (ii) $R \subset S$; (iii) $E \cap R = S$;
(iv) $E \cup I = I$; (v) $I \cup E \cup S = U$; (vi) $R \cap E = \emptyset$;
(vii) $I \cap S = \emptyset$; (viii) $(U \cap S) \cup (U \cap S') = U$.

CHAPTER XV

Graphs on the XY-Plane

There is an infinite number of solutions to the equality $x+3=y$. By giving different values to x we can calculate some values of y that make the relation $x+3 = y$ true.

x	y
0	3
1	4
2	5
3	6

(0, 3) written in round brackets is called an *ordered pair of numbers*, and represents the coordinates of a point on the XY-plane: (1, 4), (2, 5), and (3, 6) are also ordered pairs of numbers.

DOMAIN. The set of integers in the above table which represents the values we have given to x is called *the domain of the variable* x, or the domain of the relation.

In this case, we have given certain specific values to x, so that $D = \{0, 1, 2, 3\}$. But where the domain of a relation is not specifically indicated, as it is in this example, the domain is assumed to be the set of *all* real numbers.

RANGE. The set of values of y corresponding to our specific values for x is called *the range of the relation.*

$$R = \{3, 4, 5, 6\}$$

Solution Set. The set of ordered pairs,

$$S = \{(0, 3), (1, 4), (2, 5), (3, 6)\}$$

which satisfies the relation $y = x+3$ for the domain indicated is called the solution set.

Each element in S consists of *an ordered pair*, the first number being from set D, and the second the corresponding number from set R.

The relation $y = x+3$, where x has the values 0, 1, 2, or 3, can be indicated:

(*a*) by a formula: "$y = x+3$, x being 0, 1, 2, or 3";
(*b*) by a verbal description: "y is a number 3 more than x; and x is an element of the set {0, 1, 2, 3}";
(*c*) by a list of ordered pairs, thus:
$x = 0, 1, 2, 3,$
$y = 3, 4, 5, 6;$
(*d*) by a graph.

But the set of ordered pairs {(0, 3), (1, 4), (2, 5), (3, 6)} *is* the relation.

SUMMARY

In any relation involving x and y, the set of permissible values of x is known as the *domain* of the relation; the set of corresponding values of y is known as the *range* of the relation.

A *relation* is a set of ordered pairs whose first coordinates are elements of the domain of the relation, and whose second coordinates are the corresponding elements of the range.

Graphing the Relation

(*a*) The graph of the relation $y = x+3$ for the values of x (or for the domain) 0, 1, 2, 3 is a set of four isolated points.

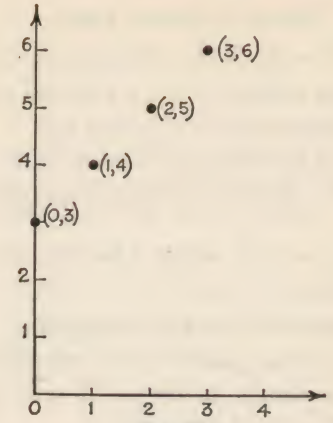


FIG. 67

In this graph, the domain of the relation is (figure 67):

$$\{x \mid x \text{ an integer, and } 0 \leqq x \leqq 3\}$$

(*b*) If the domain of the relation is extended to include *all real numbers* between 0 and 3, and including 0 and 3, the graph becomes the line AB.

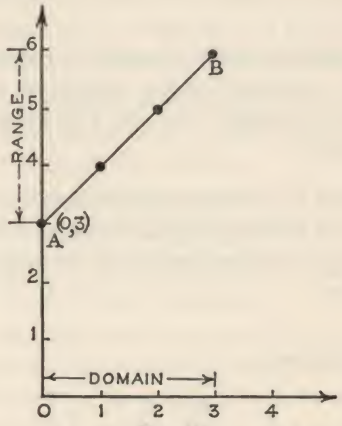


FIG. 68

In figure 68, the domain of the relation is $\{x \mid x$ a real number, and $0 \leqq x \leqq 3\}$.

The solution set of the relation $y = x+3$ where x is 0 or 1 or 2 or 3 is $\{(0, 3), (1, 4), (2, 5), (3, 6)\}$, which is a *finite* set with 4 elements. Hence its graph consists of 4 isolated points.

But the solution set of the relation $y = x+3$ where x is any real number from 0 to 3 inclusive, is an *infinite* set. Hence its graph consists of a line joining the point (0, 3) to the point (3, 6).

Since this is an infinite set of ordered pairs, we cannot write down all its elements, but here are a few listed at random:

(0·127, 3·127), (0·345, 3·345), (0·5, 3·5), (1·7, 4·7)

The set of ordered pairs in the first graph is a *subset* of the set of ordered pairs in the second.

EXERCISE 14

Plot the following pairs of sets, taking the universe in each case as the set of real numbers. Find the intersection and union of each pair.

1. $A = \{(x, y) \mid x+y = 4\}$ and $B = \{(x, y) \mid x-y = 0\}$
2. $A = \{(x, y) \mid x-y = 0\}$ and $B = \{(x, y) \mid y+x = 3\}$
3. $A = \{(x, y) \mid 2x+y = 3\}$ and $B = \{(x, y) \mid x-2y = 4\}$
4. $A = \{(x, y) \mid 2x-3y = 5\}$ and $B = \{(x, y) \mid x+2y = 6\}$
5. $A = \{(x, y) \mid 2x+3y = 7\}$ and $B = \{(x, y) \mid 4x+6y = 9\}$

CHAPTER XVI

Graphs of Inequalities (ii)

To find the solution set of the inequality $y \geqq x+3$, where the domain is $0 \leqq x \leqq 3$ or $\{x \mid x \text{ real}, 0 \leqq x \leqq 3\}$.

The solution set is indicated by the shaded portion of the graph, which is bounded by—and includes—the line AB, and is continued indefinitely upwards, as indicated by the arrows.

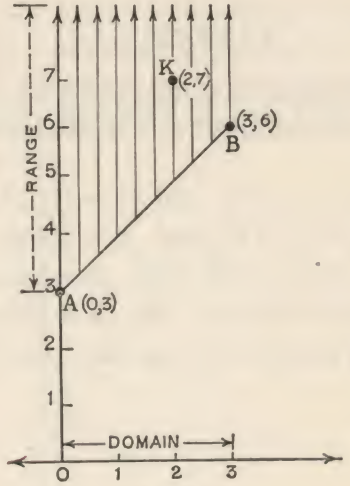


FIG. 69

Every point in this area represents an ordered pair of numbers in the solution set. The point K, for instance, represents the ordered pair (2, 7).

The line AB is included in the graph, since for every point on it, $y = x+3$, and hence each such point is an element of $y \geqq x+3$.

For the relation $y > x+3$ and the same domain, however, the line AB is *not* included.

In this case, you may draw and shade the graph as for $y \geqq x+3$, and then write: "The line AB, the graph of $y = x+3$, is not included."

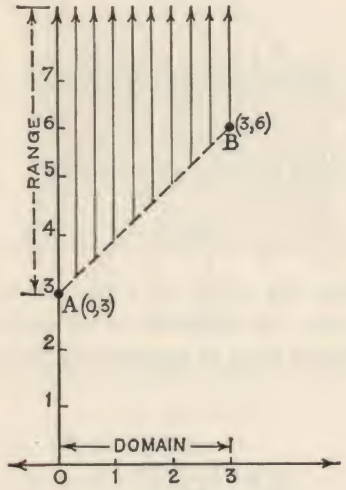


FIG. 70

Alternatively, you may indicate AB by means of a broken line. This shows that points on AB are not part of the graph, and that this line bounds but is not part of the solution set.

Graph of $y \leqq x+3$

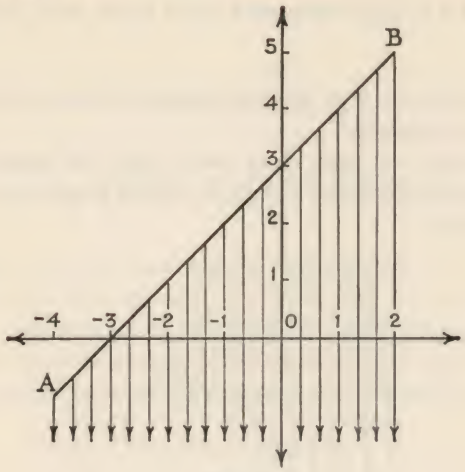


FIG. 71

Figure 71 shows the graph of $y \leqq x+3$ for the domain $-4 \leqq x \leqq 2$, x real. All points on or vertically under the line AB represent ordered pairs of numbers in the solution set.

EXERCISE 15

Draw the graph of the set of ordered pairs described in each of the following examples. In all cases, assume that x and y are real numbers.

1. Set $A = \{(x, y) \mid y = 2x, \quad -5 \leqq x \leqq 5\}$

Shade with vertical lines the set

$B = \{(x, y) \mid y \geqq 2x$, same domain as in set A$\}$

Shade with horizontal lines the set

$C = \{(x, y) \mid y \leqq 2x$, same domain as in A$\}$

2. Set $D = \{(x, y) \mid y-3x = 5$, domain $-2 \leqq x \leqq 3\}$

Shade with vertical lines the set

$E = \{(x, y) \mid y-3x > 5$, domain $-2 \leqq x \leqq 3\}$

Shade with horizontal lines the set

$F = \{(x, y) \mid y-3x < 5$, domain $-2 \leqq x \leqq 3\}$

3. Find the intersection of

$A = \{(x, y) \mid y = 2x-1\}$ and $B = \{x, y \mid 2y = 3x+1\}$

4. The set $K = \{(x, y) \mid 0 \leqq y \leqq 3, -1 \leqq x \leqq 4, x$ and y real$\}$ is shown here by shading.

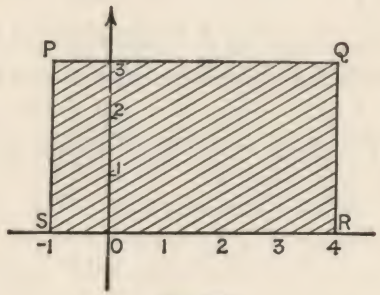


FIG. 72

Describe the set of ordered pairs for the domain

$-1 \leqq x \leqq 4$

(i) above PQ; (ii) below SR.

5. Draw the graph $\{(x, y) \mid y = x, x$ real$\}$.

On your graph indicate by horizontal shading the set

$\{(x, y) \mid y > x, x$ and y real$\}$

6. Draw the graph $\{(x, y) \mid y = -x, x$ real$\}$.

On your graph shade by vertical lines the set

$\{(x, y) \mid y < -x, x$ and y real$\}$

7. By comparing your results to questions 5 and 6, or by repeating the graphs on the same set of axes, find the intersection of the two sets $\{(x, y) \mid y > x, x$ and y real$\}$ and

$\{(x, y) \mid y < -x, x$ and y real$\}$

EXERCISE 16. REVISION

Use graphs to find the intersection of the two sets described in each example. Assume that x and y are real numbers.

1. $A = \{(x, y) \mid 3x-y = 3\}$; $B = \{(x, y) \mid x+2y = 8\}$

2. $A = \{(x, y) \mid 4x+y = 0\}$; $B = \{(x, y) \mid 5x-y = -9\}$

3. $A = \{(x, y) \mid y > x\}$; $B = \{(x, y) \mid y > -x\}$

4. $A = \{(x, y) \mid y-x > 4\}$; $B = \{(x, y) \mid y+x > 1\}$

5. $A = \{(x, y) \mid y-x > 4\}$; $B = \{(x, y) \mid y+x < 1\}$

6. $A = \{(x, y) \mid y-x \geqq 4\}$; $B = \{(x, y) \mid y+x \leqq 1\}$

7. $A = \{(x, y) \mid y-2x = -3\}$; $B = \{(x, y) \mid 2y+x+1 = 0\}$

8. $A = \{(x, y) \mid y-2x \geqq -3\}$; $B = \{(x, y) \mid 2y+x+1 \geqq 0\}$

CHAPTER XVII

Absolute Value

The distance between any two points is always positive.

If a fly starts at the point 7 on the real number line and moves to the right until it reaches the point 10, it has covered a distance of 3 units.

We might obtain this answer by asking "What must I add to 7 to arrive at 10?" or "If $7+x = 10$, what is x?"

Suppose the fly now walks back from 10 to 7. Obviously it has again covered 3 units. But if we ask our questions as before, saying "What must I add to 10 to arrive at 7?" or "If $10+x = 7$, what is x?" our answer is now -3.

But negative distance does not exist. One must move a positive distance, or not at all. The minus before our second answer indicates merely that the fly, on his return journey, travelled in the opposite direction.

To indicate distance only, *without regard to direction*, we speak of the "absolute value" of numbers; and by the absolute value of a number we mean *its positive value without regard to its sign.*

In this notation, the distance travelled by the fly, in either direction, is denoted by $|3|$, which is the symbol for the "absolute value" of both $+3$ and -3.

$7-10$ has the same absolute value as $10-7$. To express this in symbols we write: $|7-10| = |10-7| = +3$.

So, too $|5| = |-5| = 5$; $|-\sqrt{2}| = \sqrt{2}$; $|0| = 0$.

The absolute value of a number is always positive or zero.

SUMMARY

The absolute value of a number is its positive value without regard to sign. The absolute value must be positive or zero.

$|x| = x$ if x is positive.
$|x| = -x$ if x is negative.
$|x| = 0$ if x is zero.

The distance between a number and zero on the real number line is its absolute value.

Example 1

Find the solution set for $\{x \mid |x| = 3\}$. Ans. $\{+3, -3\}$.
(Either $+3$ or -3 may replace the x in $|x|$.)

Example 2

Graph on the real number line $S = \{x \mid |x| > 3\}$, where $U =$ the real numbers.

If x is >3, then $|x|$ is >3.
If x is <-3, then $|x|$ is >3.

The graph therefore consists of all numbers greater than 3 or less than -3, each of which is indicated on the real number line by an open circle.

Example 3

Graph on the real number line $S = \{x \mid |x| < -7\}$.
As the absolute value can never be negative, the solution is the empty set $\emptyset$, which cannot be graphed.

Example 4

Consider $|x+5| = 9$.

Case I: $(x+5)$ may be positive and have absolute value 9, i.e., $x+5 = 9$, or $x = 4$.

Case II: $(x+5)$ may be negative and have absolute value 9,
i.e. $-(x+5) = 9$
or $-x-5 = 9$
$-x = 14$
$x = -14$

The solution set for $\{x \mid |x+5| = 9\}$ is $\{4, -14\}$.

Inequalities

Example 1

Consider the set $\{x \mid |x+3| > 7\}$

Case I: $(x+3)$ positive: $x+3 > 7 \quad \therefore x > 4$

Case II: $(x+3)$ negative: $-x-3 > 7$
$x+3 < -7$
$\therefore x < -10$

Solution set $= \{x > 4$ or $x < -10\}$.

The graph of $\{x \mid |x+3| > 7\}$ will consist of all numbers greater than 4 and less than -10, each of which is represented on the real number line by an open circle. If we draw this graph roughly, we can check our solution set.

For a number > 4, say 5, $5+3 > 7$.

For a number <-10, say -11, $-11+3 = -8$, and $|-8| > 7$.

NOTE. The absolute value of numbers is important when considering numbers denoting velocity or force, which must have a sign attached to them to denote direction. If a velocity of 30 miles per hour in a northerly direction is denoted by $+30$, a similar velocity in the opposite direction is denoted by -30. Both, however, have the same absolute value: $|30|$ or $+30$.

EXERCISE 17

1. What is the absolute value of the following:
(*a*) -6; (*b*) $-(-4)$; (*c*) $(7-5)$; (*d*) $(5-7)$; (*e*) $-(7+10)$; (*f*) (-7×0).

2. Which of the following sentences are true?
(*a*) $|-6| < 3$; (*b*) $|-3| < |-4|$; (*c*) $|-3| \leqq |-4|$;
(*d*) $2 \not< |-3|$; (*e*) $|-7| \not< |5|$;
(*f*) $|-7| \not< |-5|$; (*g*) $|\sqrt{9}| \geqq |-3|$.

3. Graph on the real number line, where the domain of x is the real numbers, the set $\mathrm{R} = \{x \mid |x| < 4\}$.

NOTE. $x \in \mathrm{R}$ means that the domain of x is the set of real numbers.

4. Graph $\{x \mid |x+4| = 3\}$, and $x \in \mathrm{R}$.

5. Graph $\{x \mid |x+4| > 3\}$, $x \in \mathrm{R}$.

6. Graph $\{x \mid |x+4| < 3\}$, $x \in \mathrm{R}$.

7. What is the solution set of $\{x \mid |x+4| \neq 3\}$?

8. Which of the two statements $\{x \mid |x| < 7\}$ and
$\{x \mid |x| > 7\}$
has for its solution set $\{x \mid -7 < x < 7\}$?

9. Evaluate: (*a*) $|-2| - |-6|$;
(*b*) $\|-2| - |-6\|$;

10. Show that $|3+5| = |3| + |5|$,
and that $|3-5| < |3| + |-5|$.

CHAPTER XVIII

The Parabola

Examples

(1) Draw the graph of the set $\{(x, y) \mid y = x^2\}$ when the domain of the set is $\{-3, -2, -1, 0, 1, 2, 3\}$.

The set of ordered pairs to be graphed is:

$\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$, that is, the seven isolated points shown in Graph 1 (p. 96).

(2) Draw the graph of the set $\{(x, y) \mid y = x^2\}$ if the domain of the set is $\{x \mid -3 \leqq x \leqq 3, x \text{ real}\}$.

The domain now includes all the real numbers from -3 to $+3$, inclusive, and the graph is now a parabola.

The set of ordered pairs in Graph 1 is a subset of the set of ordered pairs represented by all the points on Graph 2 (p. 96).

The first set is a finite set, consisting of seven ordered pairs; the second set is infinite, since it embraces an infinite number of ordered pairs, represented by a continuous line.

(3) Draw the graph of the set $\{(x, y) \mid y > x^2\}$ if the domain of the set is $\{x \mid -3 \leqq x \leqq 3, x \text{ and } y \text{ real}\}$.

METHOD. Draw the graph $y = x^2$ (Graph 2) and shade in the interior of the parabola, between the parallel lines $x = -3$ and $x = +3$, but use a broken line for the curve itself.

Graph 1

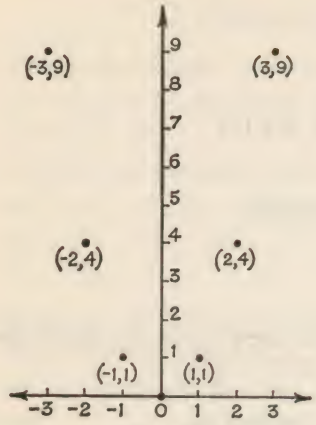


FIG. 73

Graph 2

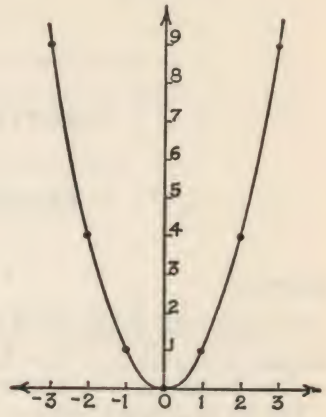


FIG. 74

For each value of x, the corresponding values of y are indicated by arrowed endless parallels to the y-axis. Thus for each value of x there is an infinite number of permissible values for y.

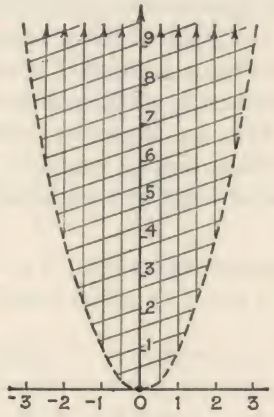


FIG. 75

NOTE: The complete graph of $y = x^2$ divides the xy-plane into three distinct regions: (i) the parabola itself; (ii) the region inside the parabola, and (iii) the region outside the parabola.

Distinction between Relation and Function

Graph 1 (figure 73) is the graph of the set $\{(x, y) \mid y = x^2\}$ where the domain is the set $\{-3, -2, -1, 0, 1, 2, 3\}$ and the range is the set $\{9, 4, 1, 0, 1, 4, 9\}$, each element in the range being the square of the corresponding element in the domain.

Each element in the domain is associated with one and only one element in the range; and a relation for which this is true is known as a *function*.

Definition. If with each element in set A there is associated in some way *one and only one* element of set B, then this relation is called "a function from A to B".

Alternatively, a relation is called a function if for every x in the domain of the relation there is one and only one corresponding y in the range of the relation.

Now let us consider the graph of $y = x^2$ (figure 74), where the domain is $\{x \mid -3 \leqq x \leqq 3, x \text{ real}\}$. Here, for the infinite number of possible values for x in the domain $-3 \leqq x \leqq +3$, there is one and only one value for x^2, and thus for y.

$\therefore$ by definition, $y = x^2$ is a function of x.

All functions are relations, but not all relations are functions.

$y^2 = x$, for instance, is not a function of x.

The reason is that each element in the domain is associated with two elements in the range and not with *one and one only*.

We can see this by considering some associated elements taken respectively from the domain and range, and by a study of the graph $y^2 = x$.

x	y
0	0
1	± 1
4	± 2
9	± 3

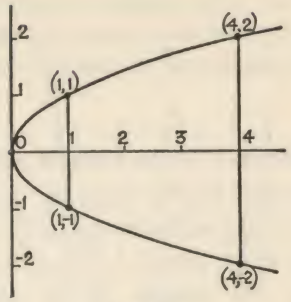


FIG. 76

By definition, $y^2 = x$ is not a function of x. (It is, however, a *relation*.) *This is a break with the traditional language of mathematics, as formerly* $y^2 = x$ *was regarded as a function of* x.

If a line parallel to the y-axis cuts the graph at more than one point, the graph represents, not a function, but a relation.

REMEMBER

All functions are relations, but not all relations are functions.

The Old Language of Mathematics and the New

Old. Draw the graph of $y = x^2-x-6$ for values of x from $x = -3$ to $x = +4$. Find from your graph the value of x when $y = 0$.

New. In the language of sets this question would read:

Draw the graph of the relation for x real:

$\{(x, y) \mid y = x^2-x-6\}$ for domain $-3 \leqq x \leqq 4$.

Find the intersection of the sets:

$\{(x, y) \mid y = x^2-x-6\}$ and $\{(x, y) \mid y = 0\}$.

EXERCISE 18

1. Find graphically the intersection of $\{(x, y) \mid y = x^2-x-6\}$ and $\{(x, y) \mid y = 0\}$.

2. Describe in set notation the three regions into which the xy-plane is divided by the relation $\{(x, y) \mid y = x^2-x-6\}$. Number the regions 1, 2, and 3, to represent respectively, (i) the curve itself; (ii) the region inside the curve; and (iii) the region outside the curve.

3. For the domain of real numbers such that $-4 \leqq x \leqq 5$, graph the relations:
$R_1 = \{(x, y) \mid y = x^2\}$ and $R_2 = \{(x, y) \mid y = x+12\}$.
Read from your graph $R_1 \cap R_2$.

4. In the graph drawn in question 3, shade with vertical lines the set or relation $V = \{(x, y) \mid y \geqq x^2$ and $y < (x+12)\}$.

5. In the resulting figure for question 4 shade with horizontal lines the set or relation $H = \{(x, y) \mid y < x^2$ and
$y > (x+12)\}$

6. On the same set of axes and with the same scales draw the graphs of $y = x^2$ and $y = -x^2$ for values of x from -4 to $+4$.

   NOTE: The second graph is the reflection of the first in the x-axis.

7. On the same set of axes and with the same scales graph the relations $A = \{(x, y) \mid y = x^2\}$ and $B = \{(x, y) \mid y^2 = x\}$ for the domain $-4 \leqq x \leqq 4$, x a real number.

8. Consider the equality $xy = 1$. If we take any value for x in this relation, the corresponding value of y can be found.

Fill in the blanks in the following table, and make a similar table for negative values of x. Then draw the graph of $xy = 1$ for values of x from $x = -4$ to $x = +4$, excluding $x = 0$.

$x =$	4	3			$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	·1	·05
$y =$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1				10	20

9. Draw the graph of $xy = -1$ for the domain of real numbers $-4 \leqq x \leqq 4$, excluding $x = 0$.

10. Draw the graph of $xy = 0$. Shade in the part of your graph that represents the inequality $xy > 0$.

CHAPTER XIX

Cartesian Products

A cross B, U cross U

Set A = {1, 2, 3}, and set B = {4, 5, 6}.

It is required to write the set of all the *ordered pairs* that can be formed by taking two elements, the first from set A and the second from set B.

Since the first element in the ordered pair may be any one of the three elements in set A, and the second can be any one of the three elements in set B, there are 3 × 3 ways in which the selection can be made.

These nine ordered pairs are:

(1, 4), (1, 5), (1, 6)
(2, 4), (2, 5), (2, 6)
(3, 4), (3, 5), (3, 6)

This set of ordered pairs is called the *Cartesian product of* A and B, and is written A × B.

A × B is a set of ordered pairs of numbers, the first taken from set A and the second from set B.

NOTE: As the word "product" is used here in a different sense from its use in arithmetic, A × B is not read as "A multiplied by B" but as "A cross B".

The Cartesian product A × B for A = {1, 2, 3} and B = {4, 5, 6} is represented by Graph 1 (figure 77), and consists of nine isolated points called "lattice" points.

By a lattice point is meant a point whose distance from each of the two axes is either zero or an integral number of units. The following are examples of lattice points: (2, 3), (0, 6), (−2, 0), (5, 7), (0, 0), (7, −3) because each of the members of these ordered pairs is either zero or an integer. The following are *not* lattice points: $(2\frac{1}{2}, 5)$, $(3\frac{1}{7}, 5\frac{2}{3})$, $(0, 3\frac{3}{4})$.

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

If $U = \{1, 2, 3, 4, 5, 6\}$ the Cartesian product $U \times U$ is represented by Graph 2 (figure 78), and consists of thirty-six isolated lattice points.

$$U \times U = \{(x, y) \mid x \in U, y \in U\}$$

Graph 1

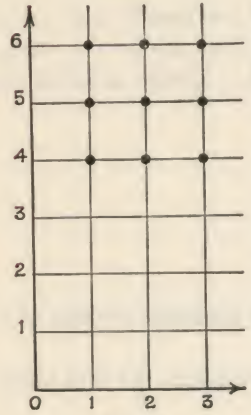


FIG. 77

Graph 2

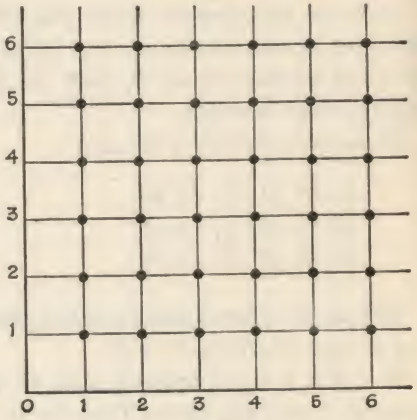


FIG. 78

Examples

1. Find the solution set of the relation $y > x-2$ where

$$x \in \{1, 2, 3\} \quad \text{and} \quad y \in \{1, 2, 3\}$$

The solution set is

$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$

x	y
1	1, 2, 3
2	1, 2, 3
3	2, 3

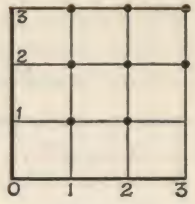


FIG. 79

2. Graph the solution set of the relation $y > x-2$, where both x and $y \in \{\text{integers}\}$.

Obviously there is an infinite number of values for x, and an infinite number of corresponding values for y.

First draw a broken line, $y = x-2$ as in figure 80. All lattice points above this line are elements of the solution set. For example:

When $x = -3, y = -4, -3, -2, -1, \ldots$
$x = -2, y = -3, -2, -1, \ldots$
$x = -1, y = -2, -1, 0, \ldots$
$x = 0, y = -1, 0, 1, \ldots$
$x = 1, y = 0, 1, 2, \ldots$

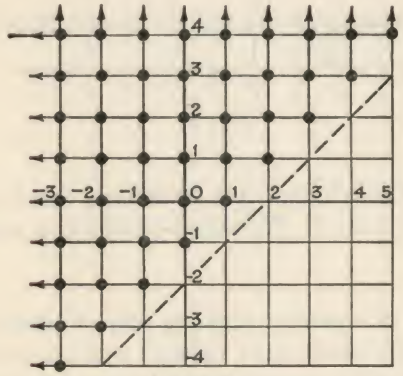


FIG. 80

Similarly, for $x = 2, 3, 4 \ldots$ and for $x = -4, -5, -6 \ldots$ The graph of $\{(x, y) \mid y > x-2, x \text{ and } y \text{ integers}\}$ is a half plane of lattice points, bounded by the line joining (2, 0) and (0, −2), and produced indefinitely in both directions.

(The solution set in *Example* 1, it will be noticed, is a subset of that in *Example* 2.)

3. Graph the solution set of the relation $y > x-2$ where x and y are any real numbers.

The solution set is the set of all ordered pairs in the half XY-plane above the broken line PQ produced indefinitely in both directions.

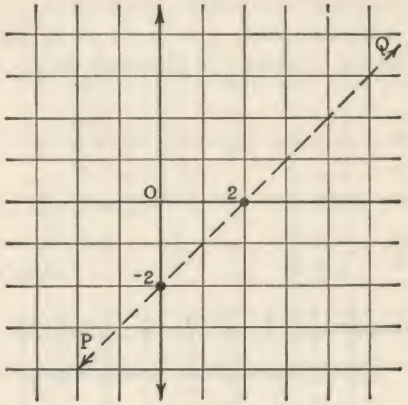


FIG. 81

PQ is drawn as a broken line, to indicate that it is not a part of the graph $y > x-2$. For points on this line, obviously, y is *equal to* $x-2$.

4. Example of a graph consisting only of lattice points.

A sum of £2 is made up of shillings and half-crowns. Draw a graph showing the possible combinations of coins.

If there are s shillings and h half-crowns, then $s+2\frac{1}{2}h = 40$, where s and h are positive integers, and where h is a multiple of 2, and $0 < h < 16$.

The possible combinations are as follows:

h	2	4	6	8	10	12	14
s	35	30	25	20	15	10	5

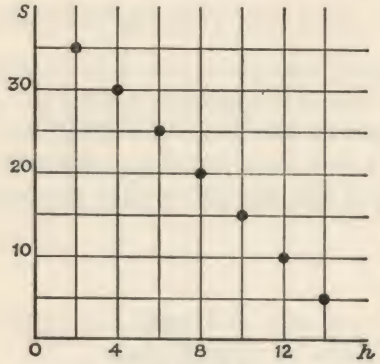


FIG. 82

The graph consists of 7 lattice points as shown.

CHAPTER XX

Abstract Proofs of Equality of Sets. De Morgan's Law. Duality

IF AND ONLY IF, OR IFF, OR ⇔

A man certainly falls into a river if he is pushed in. On the other hand he may fall in because he does not see where he is going, or because he stumbles. Accordingly, the statement "A man falls into a river *if and only if* he is pushed in", is not true.

The phrase "if and only if" is often shortened to "iff" or written ⇔.

Example

$xy = 0 \Leftrightarrow x = 0$ or $y = 0$ reads "xy equals zero, if and only if x equals zero or y equals zero".

The purpose of the double-headed arrow is to indicate that both the statement and its converse are true, or that the statement is true read from both ends.

Thus: $x = 0$ or $y = 0$ implies $xy = 0$. (Or, in symbols, $x = 0$ or $y = 0 \Rightarrow xy = 0$.) And $xy = 0$ implies $x = 0$ or $y = 0$ (or, in symbols, $xy = 0 \Rightarrow x = 0$ or $y = 0$).

On the other hand, $a = b \Rightarrow a^2 = b^2$ is not true both ways.

$a^2 = b^2$ does not imply that $a = b$, for if $a = -3$ and $b = +3$, $(-3)^2 = (+3)^2$ does not imply $(-3) = (3)$.

SUMMARY

1. $\Rightarrow$ reads "implies"; $a = b \Rightarrow a^2 = b^2$.
2. $\not\Rightarrow$ reads "does not imply"; $a^2 = b^2 \not\Rightarrow a = b$.

3. $\Leftrightarrow$ reads "if and only if"; $xy = 0 \Leftrightarrow x = 0$ or $y = 0$. AB and XY are equal chords of a circle $\Leftrightarrow$ AB and XY are equidistant from the centre.
4. A is a subset of B implies that every element of A is an element of B.
   Or, $A \subset B$ if $x \in A$ implies $x \in B$.
5. Two sets L and R are equal iff $L \subset R$ and $R \subset L$.

PROOFS FROM FIRST PRINCIPLES

Example 1

Given $A \subset B$, to prove $B' \subset A'$.

(i) Let $x \notin B$, then $x \notin A$.

(ii) Hence $x \notin B$ implies $x \notin A$, that is $x \in B'$ implies $x \in A'$.
$\therefore B' \subset A'$.

Example 2

Given $B' \subset A'$ to prove $A \subset B$.

Let $x \notin A'$ $\therefore x \notin B'$,
$\therefore x \notin A'$ implies $x \notin B'$,

that is $x \in A$ implies $x \in B$.
$\therefore A \subset B$.

In the two examples above we have proved

$A \subset B$, if and only if $B' \subset A'$;

or $A \subset B$ iff $B' \subset A'$;

or $A \subset B \Leftrightarrow B' \subset A'$.

That is, we have proved (i) that if $A \subset B$, then $B' \subset A'$; and (ii) its converse, namely, if $B' \subset A'$, then $A \subset B$.

Example 3

To prove that union is associative, that is, to prove that $A \cup (B \cup C) = (A \cup B) \cup C$.

Left side: $B \cup C = \{x \mid x \in B \text{ or } x \in C\}$

$\therefore A \cup (B \cup C) = \{x \mid x \in A \text{ or } x \in (B \cup C)\}$

$= \{x \mid x \in A \text{ or } B \text{ or } C\}$

Right side: $(A \cup B) = \{x \mid x \in A \text{ or } x \in B\}$

$\therefore (A \cup B) \cup C = \{x \mid x \in A \text{ or } B \text{ or } C\}$

$\therefore$ union is associative.

Exercise. Prove that intersection is associative, that is, prove $A \cap (B \cap C) = (A \cap B) \cap C$.

Equal Sets

Here are two equal sets, L and R:

$L = \{1, 2, 3, 4\}; \quad R = \{2, 1, 3, 4\}$

Remembering that every set is regarded as an improper subset of itself, we can say of the above sets: $L \subset R$ and $R \subset L$.

Every element of L is an element of R, and every element of R is an element of L.

We can define equal sets in several ways:

(*a*) $A = B$ if and only if $A \subset B$ and $B \subset A$; or

(*b*) $A = B$ if and only if $a \in A$ implies $a \in B$, and $b \in B$ implies $b \in A$.

The last statement can be interpreted as stating: $A = B$ if and only if every element of A is an element of B (that is, $A \subset B$) and every element of B is an element of A (that is, $B \subset A$).

To prove $(A \cap B)' = A' \cup B'$

On page 48, we illustrated by diagram in the particular case of two sets $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ that $(A \cap B)' = A' \cup B'$.

We now give a general rigorous proof from definitions.

To prove $(A \cap B)' = A' \cup B'$ we must prove (i) that the set on the left is a subset of the set on the right; (ii) that the set on the right is a subset of the set on the left; and (iii) conclude by the definition of equality at (*a*), opposite, that the set on the left equals the set on the right.

STEP I. Take the left side: $(A \cap B)'$.

Let $x \in (A \cap B)'$

then $x \notin (A \cap B)$

$\therefore x \notin A$, or $x \notin B$, or $x \notin$ either A or B

$\therefore x \in A'$, or $x \in B'$, or $x \in$ both A' and B'

$\therefore x \in (A' \cup B')$ . . . by definition of $\cup$

$\therefore x \in (A \cap B)'$ implies $x \in (A' \cup B')$

$\therefore (A \cap B)'$ is a subset of $A' \cup B'$

STEP II. Take the right side: $A' \cup B'$.

Let $x \in (A' \cup B')$

$\therefore x \in A'$ or $x \in B'$

$\therefore x \notin A$ or $x \notin B$

$\therefore x \notin (A \cap B)$

$\therefore x \in (A \cap B)'$

$\therefore$ Since $x \in (A' \cup B')$ implies $x \in (A \cap B)'$,

$A' \cup B' \subset (A \cap B)'$.

STEP III. Since $(A \cap B)' \subset A' \cup B'$

and $A' \cup B' \subset (A \cap B)'$

$\therefore (A \cap B)' = A' \cup B'$.

Q.E.D.

To prove $(A \cup B)' = A' \cap B'$

STEP I. To prove $x \in (A \cup B)'$ implies $x \in (A' \cap B')$, that is, to prove $(A \cup B)'$ a subset of $A' \cap B'$.

Let $x \in (A \cup B)'$
then $x \notin (A \cup B)$.

Therefore $x \notin A$ *and* $x \notin B$.

In other words $x \in A'$ *and* $x \in B'$,
that is, $x \in (A' \cap B')$,

$\therefore (A \cup B)' \subset (A' \cap B')$

STEP II. To prove $x \in (A' \cap B')$ implies $x \in (A \cup B)'$, that is, to prove $A' \cap B'$ is a subset of $(A \cup B)'$.

Let $x \in (A' \cap B')$
then $x \in A'$ *and* $x \in B'$;

$\therefore x \notin A$ and $x \notin B$

$\therefore x \notin (A \cup B)$.

This implies $x \in (A \cup B)'$.

Since $x \in (A' \cap B')$ implies $x \in (A \cup B)'$

$\therefore (A' \cap B') \subset (A \cup B)'$.

STEP III. Since $(A \cup B)' \subset (A' \cap B')$
and $(A' \cap B') \subset (A \cup B)'$

therefore, by definition of equality on page 108,

$$(A \cup B)' = (A' \cap B').$$

Q.E.D.

Alternative Method of Proving $(A \cup B)' = A' \cap B'$

Having proved $(A \cap B)' = A' \cup B'$ we can use complementation to establish the truth $(A \cup B)' = A' \cap B'$, as follows:

Since $(A \cap B)' = A' \cup B'$ applies to any two sets, it must apply to their complements. Accordingly we may write A′ for A, and B′ for B in $(A \cap B)' = A' \cup B'$.

Thus $(A' \cap B')' = A'' \cup B''$,

$\therefore (A' \cap B')' = A \cup B$ (since A″ is the complement of A′ and is equal to A).

Take complement of each side:

Then $(A' \cap B')'' = (A \cup B)'$
or $A' \cap B' = (A \cup B)'$.

To Prove the Distributive Law $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

STEP I. To prove L.H.S. $\subset$ R.H.S., i.e. to prove
$x \in (A \cap B) \cup C$ implies $x \in (A \cup C) \cap (B \cup C)$.

Let $x \in (A \cap B) \cup C$.

This means $x \in (A \cap B)$ or $x \in C$ (or, possibly, both). Take each possibility in turn.

If $x \in (A \cap B)$, then $x \in A$ *and* $x \in B$,

$\therefore x \in (A \cup C)$ *and* $x \in (B \cup C) \ldots$

$\therefore x \in (A \cup C) \cap (B \cup C) \ldots$

Now if $x \in C$
$x \in (A \cup C)$ *and* $x \in (B \cup C)$.

$\therefore x \in (A \cup C) \cap (B \cup C)$.

For all possibilities we have shown
$x \in (A \cap B) \cup C \Rightarrow x \in (A \cup C) \cap (B \cup C)$.

$\therefore (A \cap B) \cup C \subset (A \cup C) \cap (B \cup C)$.

STEP II. To prove R.H.S. $\subset$ L.H.S., i.e. to prove
$x \in (A \cup C) \cap (B \cup C)$ implies $x \in (A \cap B) \cup C$.

Let $x \in (A \cup C) \cap (B \cup C)$.

$\therefore x \in (A \cup C)$ *and* $x \in (B \cup C) \ldots$ (i)

Now, either $x \notin C$ or $x \in C$.

If $x \notin C$ then $x \in (A \cap B) \ldots$ from (i)

$\therefore x \in (A \cap B) \cup C$.

Even more, if $x \in C$, then $x \in (A \cap B) \cup C$.

$\therefore x \in (A \cup C) \cap (B \cup C)$ implies
$x \in (A \cap B) \cup C$.

$\therefore (A \cup C) \cap (B \cup C) \subset (A \cap B) \cup C$.

STEP III. Since $(A \cap B) \cup C \subset (A \cup C) \cap (B \cup C)$,
and $(A \cup C) \cap (B \cup C) \subset (A \cap B) \cup C$.

$\therefore (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

NOTE: An alternative method of proving equality of sets is given on page 121.

Q.E.D.

De Morgan's Law

The statement, "The complement of a union is the intersection of the complements; and the complement of an intersection is the union of the complements", known as de Morgan's Law, is important in logic. But a little effort is needed to grasp its significance.

Union consists of elements in one set *or* in another *or* in both.

Intersection consists of elements in one set *and* also in another.

Complement means *not* in a particular set.

($\cup$ implies OR, $\cap$ implies AND, and $'$ implies NOT.)

Example

"The Statue of Liberty is not' in either Ireland *or* England." This is equivalent to saying that the statue is not' in Ireland *and* not' in England.

I = set of statues in Ireland; E = set of statues in England.
$(I \cup E)'$ means, not' in either Ireland *or* England. ($\cup$, "or".)
$I' \cap E'$ means, not' in Ireland *and* not' in England. ($\cap$, "and".)

$\therefore (I \cup E)' = I' \cap E'$.

Again, to say that the metal uranium is not' mined in both Ireland *and* England is the same as saying that either it is not' mined in Ireland *or* not' in England, *or* not' in either.

$(I \cap E)'$ means, not' in both Ireland *and* England.
$I' \cup E'$ means, either not' in Ireland *or* not' in England.

$\therefore (I \cap E)' = I' \cup E'$.

EXERCISE 19

Prove from their definitions the following:

1. $A \cup B = B \cup A$
2. $A \cap B = B \cap A$
3. $(A \cap B)' = A' \cup B'$
4. $(A \cup B)' = A' \cap B'$
5. $A - B = A \cap B'$
6. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

(Hint: Use complementation, and the Distributive Law for Union proved pp. 111-12.)

Principle of Duality

$\cup$ and $\cap$ (union and intersection) are called dual operations. The principle of duality, stated here without proof, is that if a theorem involving these operations is true, the theorem obtained by substituting $\cap$ for $\cup$ and $\cup$ for $\cap$, called its *dual*, is also true. Thus the dual of $(A \cap B)' = A' \cup B'$ (proved on p. 109) is $(A \cup B)' = A' \cap B'$ (proved on p. 110).

Similarly the dual of

$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ . . . proof, page 111
is $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

The dual of $A \cup B = B \cup A$ is $A \cap B = B \cap A$.

The empty set $\emptyset$ is replaced by the universal set U when applying the duality principle to operations with sets, thus:

The dual of $A \cup (A \cap B)' = U$ is

$$A \cap (A \cup B)' = \emptyset$$

Proof that $A \cup (A \cap B)' = U$.

$$\begin{aligned} (A \cap B)' &= A' \cup B' \\ \therefore A \cup (A \cap B)' &= A \cup (A' \cup B') \\ &= (A \cup A') \cup B' \ldots \text{Associative Law} \\ &= U \cup B' = U \end{aligned}$$

Proof that $A \cap (A \cup B)' = \emptyset$.

$$\begin{aligned} (A \cup B)' &= A' \cap B' \\ \therefore A \cap (A \cup B)' &= A \cap (A' \cap B') \\ &= (A \cap A') \cap B' \ldots \text{Associative Law} \\ &= \emptyset \cap B' = \emptyset \end{aligned}$$

CHAPTER XXI

Laws Governing Operations with Sets

In this section, the laws governing operations with sets are collected for reference. The truth of those not already given in the text is obvious from Venn diagrams.

The numbering is arbitrary, and the letters *a* and *b* are used to indicate duals. Thus, 1*a* and 1*b* are duals.

Identity Laws

1*a*. $A \cup U = U$ 1*b*. $A \cap \emptyset = \emptyset$

2*a*. $A \cup \emptyset = A$ 2*b*. $A \cap U = A$

Absorption Laws

3*a*. $A \cup (A \cap B) = A$ 3*b*. $A \cap (A \cup B) = A$

Complement Laws

4*a*. $A \cup A' = U$ 4*b*. $A \cap A' = \emptyset$

5*a*. $(A')' = A$ (no dual) 5*b*. $U' = \emptyset$; $\emptyset' = U$

Commutative Laws

6*a*. $A \cup B = B \cup A$ 6*b*. $A \cap B = B \cap A$

Associative Laws

7*a*. $A \cup (B \cup C) = (A \cup B) \cup C$

7*b*. $A \cap (B \cap C) = (A \cap B) \cap C$

Distributive Laws

8*a*. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

8*b*. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's Laws

9*a*. $(A \cup B)' = A' \cap B'$ 9*b*. $(A \cap B)' = A' \cup B'$

Idempotent Laws

10*a*. $A \cup A = A$ 10*b*. $A \cap A = A$

These laws may be used in simplifying, expanding, and factorizing expressions in the algebra of sets.

CHAPTER XXII

Factorization and Expansion of Set Polynomials

In an algebraic expression such as $5xy-7yz+3abc$, the parts of the expression separated by the + and the − signs are called *terms*; thus $5xy$, $7yz$, and $3abc$ are terms. An expression with more than one term is called a *polynomial.*

In a set expression, the parts of the expression separated by $\cup$ are called terms, $\cup$ in set operations being somewhat similar to + in arithmetic or in an algebra based on numbers.

$(A \cap B) \cup (A \cap B') \cup C$ is a set polynomial with the three terms: $A \cap B$; $A \cap B'$; and C.

$\cap$ in set operations being somewhat similar to multiplication in ordinary algebra, $A \cap B \cap C$ is one term, just as *abc* is one term.

As in algebra, an expression enclosed in brackets is regarded as one term; thus

$$(A \cup B) \cup (C \cup D) \cup (A \cap B') \cup C'$$

consists of four terms.

If a polynomial is written as the intersection of sets, the sets on either side of the $\cap$ sign are called *factors.*

Example 1

Factorize $(A \cap B) \cup (A \cap C)$

$(A \cap B) \cup (A \cap C)$, which is analogous to $ab+ac$, is factorized as $A \cap (B \cup C)$, which is analogous to $a(b+c)$.

Example 2

Factorize $A \cup (B' \cap C')$

$A \cup (B' \cap C') = (A \cup B') \cap (A \cup C')$

Explanation: The $A \cup$ outside the bracket is distributed over the B' and C' inside the bracket (Distributive Law 8*a*).

Example 3

To prove the Absorption Laws, that is, to prove that

$$A \cup (A \cap B) = A \cap (A \cup B) = A$$

L.H.S.: $A \cup (A \cap B)$

$= (A \cap U) \cup (A \cap B)$
$= A \cap (U \cup B)$
$= A \cap U$
$= A$

Explanation. In the top line, $A \cup$ is outside the bracket, and $A \cap$ inside.

In order to have $A \cap$ common, A is written $A \cap U$, by Identity Law 2*b*.

R.H.S.: $A \cap (A \cup B)$

$= (A \cup \emptyset) \cap (A \cup B)$
$= A \cup (\emptyset \cap B)$
$= A \cup \emptyset$
$= A$

Explanation. In the top line, $A \cap$ is outside the bracket, and $A \cup$ inside.

In order to have $A \cup$ common, A is written $A \cup \emptyset$, by Identity Law 2*a*.

Example 4

Factorize $(A \cap C) \cup (B \cap C) \cup (A \cap D) \cup (B \cap D)$

$(A \cap C) \cup (B \cap C) \cup (A \cap D) \cup (B \cap D)$ 1
$= [C \cap (A \cup B)] \cup [D \cap (A \cup B)]$ 2
$= (A \cup B) \cap (C \cup D)$ 3

Explanation. Compare the lines marked 1, 2, and 3 in the following with the corresponding lines above:

$ac+bc+ad+bd$ 1
$= [c(a+b)]+[d(a+b)]$ 2
$= (a+b)(c+d)$ 3

REMEMBER: *Union behaves somewhat like addition.*
Intersection behaves somewhat like multiplication.

Example 5

Expand $(A' \cup B) \cap (C \cup D)$.

NOTE: To expand means to express as a series of unions.

$(A' \cup B) \cap (C \cup D)$
$= [(A' \cup B) \cap C] \cup [(A' \cup B) \cap D]$
$= (A' \cap C) \cup (B \cap C) \cup (A' \cap D) \cup (B \cap D)$

Compare with

$(a'+b)(c+d) = [(a'+b)c]+[(a'+b)d] = a'c+bc+a'd+bd$

Example 6

Simplify $(A \cap B') \cup (A \cap B) \cup (A' \cap B)$

$(A \cap B') \cup (A \cap B) \cup (A' \cap B)$
$= [A \cap (B' \cup B)] \cup (A' \cap B)$
$= [A \cap U] \cup (A' \cap B)$
$= A \cup (A' \cap B)$
$= (A \cup A') \cap (A \cup B)$
$= U \cap (A \cup B)$
$= A \cup B$

EXERCISE 20

1. Write the duals of:

(*a*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(*b*) $A \cup U = U$
(*c*) $A \cap U = A$
(*d*) $A \cup (A \cap B) = A$
(*e*) $(A \cap B)' = A' \cup B'$

2. Factorize and simplify each of the following set polynomials, and check each answer by a Venn diagram:

(*a*) $(A \cap B') \cup (A \cap B)$
(*b*) $A \cup (B \cap C)$
(*c*) $(A' \cap B') \cup (A' \cap B)$
(*d*) $(B \cap A') \cup (A \cup B)'$
(*e*) $(A' \cap C) \cup (B \cap C) \cup (A' \cap D) \cup (B \cap D)$
[*No diagrams required for* (*e*)]

3. Simplify, where A and B are subsets of U. Check each result by a Venn diagram.

(*a*) $(A \cap B) \cup (A \cap B')$
(*b*) $A \cup (A \cap B)$
(*c*) $A \cap (A \cup B)$
(*d*) $B \cup B' \cup A$
(*e*) $B \cap B' \cap A$
(*f*) $(A' \cap B') \cup B$

CHAPTER XXIII

Equality of Sets Proved by Membership Tables

A universe U has subsets A and B.

If x is any element of U, four possibilities arise concerning A and B. The element can:

1. Belong to A and belong to B: $x \in A$, $x \in B$.
2. Belong to A but NOT to B; $x \in A$, $x \notin B$.
3. NOT belong to A but belong to B: $x \notin A$, $x \in B$.
4. NOT belong to A and NOT belong to B: $x \notin A$, $x \notin B$.

If we want to prove $(A \cup B)' = A' \cap B'$ we can build up a membership table as follows:

	I	II	III	IV
	A	B	$A \cup B$	$(A \cup B)'$
1.	$\in$	$\in$	$\in$	—
2.	$\in$	—	$\in$	—
3.	—	$\in$	$\in$	—
4.	—	—	—	$\in$

Explanation

The first two columns state in tabular form the four possibilities mentioned above, the dashes under A and B denoting $\notin$.

[$\notin$ *is understood in the spaces marked* —.]

Column III, A ∪ B, is deduced from columns I and II, since if x is an element of either A or B it is an element of their union.

Column IV $(A \cup B)'$ is the negation of column III.

We continue the table for A′ and B′, thus:

	V	VI	VII
	A′	B′	A′ ∩ B′
1.	—	—	—
2.	—	∈	—
3.	∈	—	—
4.	∈	∈	∈

Explanation

Columns V (A′) and VI (B′) are the negations of columns I and II.

Column VII is deduced from columns V and VI, since, to be an element of an intersection, x must be an element of both.

Finally the similarity between columns IV and VII proves the proposition $(A \cup B)' = A' \cap B'$, as it demonstrates $x \in (A \cup B)'$ implies $x \in A' \cap B'$; and $x \notin (A \cup B)'$ implies $x \notin A' \cap B'$.

	IV	VII
	$(A \cup B)'$	$A' \cap B'$
1.	$\notin$	$\notin$
2.	$\notin$	$\notin$
3.	$\notin$	$\notin$
4.	∈	∈

Example

To prove by membership tables that:

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

With three subsets A, B, and C in a universe U, any element x of U may be in any of the eight disjoint regions into which, in a Venn diagram, three overlapping circles divide the universe. See figure 12, page 31.

Left Side

	I	II	III	IV	V
	A	B	C	A ∩ B	(A ∩ B) ∪ C
1.	∈	∈	∈	∈	∈
2.	∈	∈	—	∈	∈
3.	—	∈	∈	—	∈
4.	∈	—	∈	—	∈
5.	—	∈	—	—	—
6.	∈	—	—	—	—
7.	—	—	∈	—	∈
8.	—	—	—	—	—

Right Side

	VI	VII	VIII
	$A \cup C$	$B \cup C$	$(A \cup C) \cap (B \cup C)$
1.	$\in$	$\in$	$\in$
2.	$\in$	$\in$	$\in$
3.	$\in$	$\in$	$\in$
4.	$\in$	$\in$	$\in$
5.	—	$\in$	—
6.	$\in$	—	—
7.	$\in$	$\in$	$\in$
8.	—	—	—

Columns V and VIII are the same, thus proving the proposition.

EXERCISE 21

Construct membership tables for a universe U with subsets A and B to prove the following:

1. $A \cup B = B \cup A$

2. $A \cup (A \cap B) = A$

3. $(A' \cup B')' = A \cap B$

4. $(A \cap B)' = A' \cup B'$

5. $A \cap (A \cap B)' = A \cap B'$

Note

We have considered three formal definitions of equality of sets. Thus:

Two sets, L and R are equal:

(i) If $L \subset R$ and $R \subset L$;
(ii) If $l \in L$ implies $l \in R$, and $r \in R$ implies $r \in L$;
(iii) If $x \in L$ implies $x \in R$, and $\{y \mid y \in R, y \notin L\} = \emptyset$

These are equivalent definitions. That (ii) is the same as (i) can be seen from the definition of subset; and (iii) is another expression of (ii), and so of (i), as it means that any element of L is an element of R, *and there is no element of R which is not in L.*

Membership tables are a comparatively simple way of seeing whether sets are equal. They are based mainly on (ii) and (iii). The tables exhaust *all* the possibilities. We end with x an element of the left side of an equation if and only if it is an element of the right side. The third formula emphasizes the "*only* if" aspect.

However, in this book we have concentrated mainly on (i) rather than on membership tables, as (i) is the proof used most often in dealing with the equivalence of infinite (or "transfinite") sets.

(Tables similar to membership tables, called truth tables, are commonly used in mathematical logic to check the truth or falsehood of compound statements.

The system of applying the laws of algebra to logic was introduced about a century ago by George Boole, of whom we speak in the next chapter.)

CHAPTER XXIV

Boolean Algebra with Alternative Notation

The algebra we have been studying is known as a Boolean algebra, after its inventor, George Boole (1815–1864), a brilliant English logician and mathematician. Although he never studied at a university, being taught by his father, who was a tradesman, Boole was appointed to the Chair of Mathematics in Cork University at the age of 34. He is famous for his *Laws of Thought*, in which sets of objects or ideas denoted by mathematical symbols are treated by the laws of algebra so that logical results are obtained by mathematical processes.

Owing to the practical applications of Boolean algebra in electrical circuit diagrams, discovered as late as 1937, and in the exceedingly complex science of cybernetics (the word was coined in 1947) the subject has recently had widespread study. Mathematicians are not agreed as to the best symbols to use in set algebra. The modern tendency is to use $+$ for $\cup$ and $\times$ for $\cap$, and the symbol a', as before, for the complement of a.

Thus $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ now becomes

$$a\,(b+c) = ab+ac$$

This notation has the advantage that many of the rules for expansion and factorization appear the same in both algebras; there are, however, cases in which they differ.

Two further modifications are needed to bring Boolean algebra into line with ordinary numerical algebra: namely, the substitution of the symbol 1 for the universal set U, and of 0 for the null set $\emptyset$. With the two symbols 1 and 0 (the only

symbols used by the computer in its mighty calculations) and the symbols $+$, $\times$, and $'$, all the operations of set algebra can be described. It is important to remember that this 1 and 0 are NOT the familiar "one" and "zero" of arithmetic, but are the universal and null sets, respectively.

Here let us restate the principles of duality in both notations.

(i) $\cup$ and $\cap$ are dual operations, and if a statement involving these is true, then its dual (that is, a statement where $\cap$ is substituted for $\cup$, and $\cup$ for $\cap$) is also true; the duality principle requires the interchange of U and $\emptyset$.

(ii) $+$ and $\times$ are dual operations, and if a statement involving these is true, its dual (that is, a statement where $\times$ is substituted for $+$, and $+$ for $\times$) is also true; the duality principle requires the interchange of 1 and 0.

The laws on pp. 115-16 in $\cup$ and $\cap$ notation are here collected for reference in $+$ and $\times$ (or .) notation.

Identity Laws

1a. $a+1 = 1$	1b. $a\,.\,0 = 0$
2a. $a+0 = a$	2b. $a\,.\,1 = a$

Absorption Laws

3a. $a+ab = a$	3b. $a(a+b) = a$

Complement Laws

4a. $a+a' = 1$	4b. $a\,.\,a' = 0$
5a. $(a')' = a$ (no dual)	5b. $0' = 1$; $1' = 0$

Commutative Laws

6a. $a+b = b+a$	6b. $ab = ba$

Associative Laws

7a. $(a+b)+c = a+(b+c)$	7b. $(ab)c = a(bc)$

Distributive Laws

8a. $a+bc = (a+b)(a+c)$ 8b. $a(b+c) = ab+ac$

De Morgan's Laws

9a. $(a+b)' = a'b'$ 9b. $(ab)' = a'+b'$

Idempotent Laws

10a. $a+a = a$ 10b. $a \,.\, a = a$

EXERCISE 22

By referring to the laws given above, say whether the following equations are (*a*) true in both ordinary and Boolean algebra; or (*b*) true only in Boolean algebra; or (*c*) true only in ordinary algebra:

1. $a+0 = a$ **2.** $a+1 = 1$ **3.** $a \,.\, 1 = a$
4. $a \,.\, a = a$ **5.** $a \,.\, a = a^2$ **6.** $a+a' = 1$
7. $a \,.\, a' = 0$ **8.** $a(b+c) = ab+ac$
9. $a+bc = (a+b)(a+c)$ **10.** $(a')' = a$
11. $a \,.\, 0 = 0$ **12.** $0' = 1$ **13.** $1+1 = 1$
14. $1+0 = 1$

EXERCISE 23

Write the fourteen statements given in Exercise 22 in the other (that is, the $\cap$ and $\cup$) notation.

Abstract Laws of Set Operations

The list on pp. 127-8, similar to that on pp. 115-16, states the laws governing the set operations of union, intersection, and complementation. These are the basic laws of Boolean algebra, and any system obeying such laws is called a Boolean algebra.

The letters used in the laws need not necessarily refer to sets. and the operations need not be set operations. It is agreed that $+$ or $\cup$ stands for *or*; that $\times$ or $\cap$ stands for *and*; that 0 stands for *false*; and that 1 stands for *true*.

Now if the letter *a* stands for the statement "The day is fine", then the letter a' stands for the statement "The day is not fine".

$a+a' = 1$ is the true statement "The day is fine, or else the day is not fine".

$a \,.\, a' = 0$ labels the statement "The day is fine and the day is not fine" as false.

$a \,.\, a$, which is equivalent to *a*, is the emphatic though tautological statement "The day is fine and the day is fine", which adds nothing but emphasis to the statement "The day is fine".

CHAPTER XXV

Identity Elements—Boolean Algebra of Sets—Expansion and Factorization in Two Notations—Extension of Complement Laws

Identity Elements

Reason for substituting 1 for U and 0 for $\emptyset$ in the Boolean algebra of sets:

Because any number multiplied by 1 gives as a result an identical number:

1 *is called the identity element under the operation of multiplication.*

	$\times$		$\cap$
Compare	$1 \cdot 1 = 1$	with	$A \cap U = A$
	$2 \cdot 1 = 2$	with	$B \cap U = B$
	$3 \cdot 1 = 3$	with	$C \cap U = C$, etc.

1 is the identity element under the operation $\times$.

U is the identity element under the operation $\cap$.

Hence the substitution of 1 for U in the operation $\cap$ since U with $\cap$ behaves like 1 with $\times$.

Similarly, 0 is the identity element for addition of real numbers:

	$+$		$\cup$
Compare	$1+0 = 1$	with	$A \cup \emptyset = A$;
	$2+0 = 2$	with	$B \cup \emptyset = B$; etc.

Since $\emptyset$ under the operation $\cup$ behaves like 0 under the operation $+$, $\emptyset$ is the identity element with $\cup$.

For skill in working with more complicated expressions in set algebra it is desirable to have some practice in Boolean algebra. We shall work some examples, remembering that the right side of the frame below contains some Boolean algebra that has no counterpart in numerical algebra.

1 corresponds to U
0 corresponds to $\emptyset$
$+$ corresponds to $\cup$
$\times$ corresponds to $\cap$

$\cup$	$+$
$A \cup A = A$	$a+a = a$
$A \cup A' = U$	$a+a' = 1$
$A \cup U = U$	$a+1 = 1$
$A \cup \emptyset = A$	$a+0 = a$
$\cap$	$\times$
$A \cap A = A$	$a \cdot a = a$
$A \cap A' = \emptyset$	$a \cdot a' = 0$
$A \cap U = A$	$a \cdot 1 = a$
$A \cap \emptyset = \emptyset$	$a \cdot 0 = 0$

Example 1

Simplify $A \cap (B \cup A)$

$a(b+a)$

$= ab+a \cdot a$

$= a(b+1)$

$= a$, since $b+1 = 1$, and $a \cdot a = a$ (above)

$\therefore A \cap (B \cup A) = A$ (Absorption Law)

Example 2

The dual of this, also an absorption law, is

$$A \cup (B \cap A) = A$$

and is proved thus:

$$a+ba = a(1+b) = a, \text{ since } 1+b = 1$$

NOTE: ba is only a subset of a, and therefore has no extra elements to contribute to a when the union $a+ba$ is formed: ba is said to be *absorbed* into a. A Venn diagram will make this clear.

The Absorption Law is useful in simplifying expressions; it applies when a term appears which is merely the repetition of another term with some extra factor, thus: $a+ab = a$. The term ab is a repetition of a with the extra factor b: ab is absorbed into a. Again $a+ab+abc = a$.

Here abc is a repetition of ab with the extra factor c; abc is absorbed into ab, which in turn is absorbed into a. A Venn diagram will clarify the idea that abc is only a subset of ab, and show that the union of abc with ab gives ab. As before, the union $ab+a$ gives a.

Example 3

Prove that $(A \cap B) \cup (A \cap B') = A$

Proof. $ab+ab' = a(b+b') = a \,.\, 1 = a$

NOTE: A mental image of the fact that

$$(A \cap B) \cup (A \cap B') = A$$

comes from the statement: "The set of pupils in class A taking Botany ($\cap$ B) when united with those in A not taking Botany ($\cap$ B′) makes up the whole class A."

Example 4

Simplify

$$[(A \cap B') \cup (B \cap C)] \cap [(A \cap C') \cup (B' \cap C')]$$

Writing in ordinary + and × notation we get:

$$\begin{aligned}&(ab'+bc)(ac'+b'c')\\ &= aab'c'+abcc'+ab'b'c'+bcb'c'\\ &= aab'c'+ab(0)+ab'b'c'+0\\ &= ab'c'+0+ab'c'+0\\ &= ab'c'+ab'c' = ab'c',\end{aligned}$$

or $A \cap B' \cap C'$

EXERCISE 24

Simplify the following expressions in Boolean algebra:

1. aa' **2.** $a+a'$ **3.** $1+a$ **4.** $a+ab$
5. $a+ab+abc$ **6.** $abc+abc'$ **7.** $a(a'+b)$
8. $ab(a'+b')$ **9.** $(a+b')(a'+b)$
10. $abc+abc'+ab'c'+ab'c$

Factors

Factors such as $ab+ac=a(b+c)$ are the same in both algebras.

Set algebra, however, has two laws of distributivity: not only is multiplication distributive over addition, as in

$$a(b+c) = ab+ac$$

but also *addition is distributive over multiplication.* Thus we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, which can be written in Boolean algebra in the quite unfamiliar form $a+bc = (a+b)(a+c)$, i.e. the a is distributed over the b and c.

(This concept may be difficult for students who are weak at ordinary factors.)

Here are some examples:

(i) $a+bcd = (a+b)(a+c)(a+d)$
(ii) $ab+cd = (ab+c)(ab+d)$
$= (a+c)(b+c)(a+d)(b+d)$

RULE

To factorize two terms joined by +(plus) we form all possible additions of letters, one letter from each term, and multiply them together.

Example

From $a+bc$ it is possible to form two additions of letters, one from each term, namely $(a+b)$ and $(a+c)$. These two multiplied together give the factors of $a+bc$. Thus $a+bc = (a+b)(a+c)$.

Similarly, as in example (i) on pp. 133, from $a+bcd$ it is possible to form three additions of letters, one from each term, namely $(a+b)$, $(a+c)$, and $(a+d)$. The product of these three, that is, $(a+b)(a+c)(a+d)$, gives the factors in Boolean algebra of $a+bcd$; in short, a is *distributed* over b and c and d.

EXERCISE 25

Factorize and simplify the following:

1. $ab+ac$ **2.** $ab+abc$
3. $ab+ab'$ **4.** $abc+abc'$
5. $abc+abc'+abd$ **6.** $ac+ad+bc+bd$

Factorize and simplify by working first in + and × notation:

7. $(A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$
8. $(A \cap B \cap C) \cup (A \cap B \cap C')$
9. $(A' \cap B') \cup (B' \cap C) \cup (A' \cap D') \cup (C \cap D')$
10. $(A' \cup B) \cup (A \cup B')$

Before attempting the following, students are advised to read carefully the rule given above, and to remember that in set algebra ADDITION DISTRIBUTES OVER MULTIPLICATION.

Factorize:

11. $a+bc$ **12.** $a+bcd$
13. $ab+cdf$ **14.** $a'b+ab'$
15. $ab+a'b'$

Complements

Here we revise and extend the de Morgan laws in both notations.

$$(A \cup B)' = A' \cap B', \text{ corresponding to } (a+b)' = a'b'$$
$$(A \cap B)' = A' \cup B', \text{ corresponding to } (ab)' = a'+b'$$
$$(A \cup B \cup C)' = [(A \cup B) \cup C]' = (A \cup B)' \cap C'$$
$$= A' \cap B' \cap C';$$

or,

$$(a+b+c)' = [(a+b)+c]' = (a+b)'c' = a'b'c'$$

Similarly,

$$(A \cup B \cup C \cup \ldots \cup N)' = A' \cap B' \cap C' \ldots \cap N';$$

or, $(a+b+c \ldots +n)' = a'b'c' \ldots n'.$

Corresponding to the extension of $\cup$ we have the extension of $\cap$. Thus

$$(A \cap B \cap C \ldots \cap N)' = A' \cup B' \cup C' \ldots \cup N';$$

or, $(abc \ldots n)' = a'+b'+c' \ldots +n'.$

RULE

To write the complement of any expression E, (i) interchange the symbols $\cup$ and $\cap$ (+ and ×), and (ii) take the complement of each term or factor in E (a' for a, a for a').

Examples

(i) $(a'+b+c)' = ab'c'$

(ii) $(ab+c')' = (ab)'c = (a'+b')c$

(iii) $(a'b+b'c+c'a)' = (a'b)'(b'c)'(c'a)'$
$= (a+b')(b+c')(c+a')$

(iv) $[a'+(a+b)]' = (a')' \,.\, (a+b)' = a \,.\, (a'b') = aa'b = 0$
or $[a'+(a+b)]' = [a'+a+b]' = [1+b]' = [1]' = 0$

(v) $[a'+ab]' = (a')' \,.\, (ab)' = a \,.\, (a'+b') = aa'+ab' = ab'$

(WARNING. Remember that $(ab)' = a'+b'$ and NOT $a'b'$. See example (v) above.)

(vi) $[ab'+a'b]' = (ab')' \,.\, (a'b)'$
$= (a'+b)(a+b')$
$= a'a+ab+a'b'+bb'$
$= 0+ab+a'b'+0$
$= ab+a'b'$

(vii) Simplify

$\{A \cap [(B' \cap C) \cup (B \cap C')]'\} \cup \{A' \cap [(B' \cap C) \cup (B \cap C')]\}$

Working in + notation we get

$\{a \times [b'c+bc']'\}+\{a' \times [b'c+bc']\}$
$= a[(b'c)' \,.\, (bc')']+a'b'c+a'bc'$
$= a[(b+c')(b'+c)]+a'b'c+a'bc'$
$= a[bb'+b'c'+bc+cc']+a'b'c+a'bc'$
$= ab'c'+abc+a'b'c+a'bc'$ (since $bb' = cc' = 0$)

Result:

$(A \cap B \cap C) \cup (A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)$

EXERCISE 26

Prove the following identities:

1. $(A' \cap B)' = A \cup B'$
2. $(A \cup B')' = A' \cap B$
3. $A \cap (A \cap B)' = A \cap B'$
4. $(A' \cup B')' \cup (A' \cup B)' = A$
5. $(A' \cup B \cup C)' = A \cap B' \cap C'$
6. $[(A \cap B) \cup C']' = (A' \cup B') \cap C$
7. $[(A' \cap B) \cup (B' \cap C) \cup (C' \cap A)]'$
$= (A \cup B') \cap (B \cup C') \cap (C \cup A')$
8. $[A' \cup (A \cup B)']' = A$
9. $[(A \cap B') \cup (A' \cap B)]' = (A' \cap B') \cup (A \cap B)$
10. $(A \cap B \cap C) \cup (A' \cap B \cap C) \cup (A \cap B' \cap C)$
$\cup (A \cap B \cap C') = (A \cap B) \cup (B \cap C) \cup (C \cap A)$

SYMBOLS USED IN SET NOTATION

1. **Set.** A set is a collection of clearly defined objects, which are known as the *members* or *elements* of the set. $\in$ reads "is an element of".
2. **Subset.** A subset is a set that is included in another set. $\subset$ reads "is a subset of".
   NOTE.—*Every set is regarded as a subset of itself.*

3. $\emptyset$ or $\{\ \}$ is the **Empty** or **Null Set.**

4. **Proper Subset.** B is a proper subset of A if it is a subset which does not contain all the elements of A.
   NOTE: $\emptyset$ is a subset of every set.
5. **Equal Sets.** Two sets are equal if they contain exactly the same elements.
6. **Equivalent Sets.** Two sets are equivalent if the elements of each can be put into one-to-one correspondence with the elements of the other.
   $\sim$ reads "is equivalent to".
7. U, the **Universal Set**, or the **Universe**, means the complete body of elements under discussion.

8. **Complement.** A′ or $\bar{A}$ is the set of all elements of the universe not in A.

9. $\cap$ denotes the intersection of two sets. $A \cap B$ = intersection of A and B.

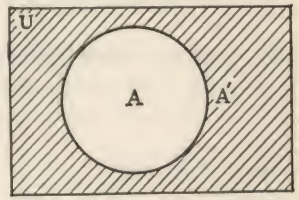
10. $\cup$ denotes the union of two sets. $A \cup B$ = union A and B.

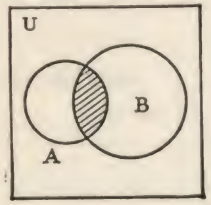
1. $A = \{1, 3, 5, 7\}$. A is the set of the first four odd integers. $5 \in \{1, 3, 5, 7\}$

2. $B = \{1, 3\}$ is a subset of A (above) because it is included in A. $B \subset A$. B is a subset of A if and only if every element of B is an element of A.
   $\{a, b, c, d\}$ is a subset of $\{a, b, c, d\}$. $A \subset A$.
3. $\emptyset$ is the set with no elements. It is *not* $\{0\}$, which contains one element, namely 0.
4. $\{a, b, c\}$ is a proper subset of $\{a, b, c, d\}$.
   $\{\ \}$ is a subset of all sets.

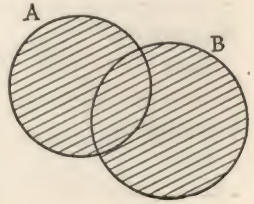
5. $H = \{1, 2, 3\}$; $K = \{1, 2, 1, 3\}$. $H = K$. $H \subset K$, and $K \subset H$. (An element is counted once only.)
6. $R = \{1, 2, 3\}$. R and S are equivalent.
   $\updownarrow\ \updownarrow\ \updownarrow$
   $S = \{a, b, c\}$. $R \sim S$.

7. When the universe is all the positive numbers 1 to 9, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
8. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{4, 5, 6\}$, then $A' = \{1, 2, 3, 7, 8, 9\}$.



9. 

$A \cap B$ is shaded. $A \cap B$ is the set of elements common to A and B.



10. $A \cup B$ is shaded.
    $A \cup B$ is the set of elements either in A, *or* in B, *or* in both A and B.

11. **Complement of Intersection,** $(A \cap B)'$, in universe U, is the set of all elements *not* in $A \cap B$.

$$(A \cap B)' \cup (A \cap B) = U$$

12. **Complement of Union,** $(A \cup B)'$, in universe U, is the set of all elements not in $A \cup B$.

13. $\Leftrightarrow$ means "if and only if", "iff".
14. $\Rightarrow$ reads "implies".

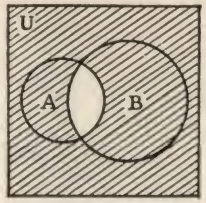
15. **Cartesian-Product Set.** $A \times B$ is the set of all possible ordered pairs of the elements of A and B, the first element chosen from A, and the second from B.
16. **Cartesian-Product Set.** $A \times A$. (See opposite.)

17. $U \times U$. (See opposite.)

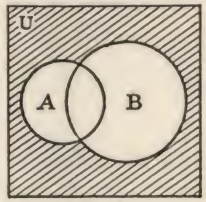
Laws of Algebra

	Addition	Multiplication
Commutative Law	$a+b = b+a$	$ab = ba$
Associative Law	$(a+b)+c = a+(b+c)$	$(ab)c = a(bc)$
Distributive Law		$a(b+c) = ab+ac$

11. $(A \cap B)'$ is shaded.
Compare with No. 9.



12. $(A \cup B)'$ is shaded.
Compare with No. 10.



13. $xy = 0 \Leftrightarrow x = 0$ or $y = 0$; $\Leftrightarrow$ reads both ways.
14. British $\Rightarrow$ European. But European $\not\Rightarrow$ British.
$\Rightarrow$ reads only one way, as indicated by arrow-head.
15. $A = \{1, 2\}$; $B = \{5, 6\}$. $A \times B$ (read "A cross B") $= \{(1, 5), (1, 6), (2, 5), (2, 6)\}$

16. A cross A is the set of all possible ordered pairs whose co-ordinates are the members of A.
17. U cross U is the set of all possible ordered pairs (x, y) such that x is an element of U and y is an element of U. U cross U is the whole XY-plane, when U is the set of real numbers.

Commutative Law

$A \cup B = B \cup A.$ $\qquad$ $A \cap B = B \cap A.$

Associative Law

$(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Law

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, or $\cup$ distributes over $\cap$.
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, or $\cap$ distributes over $\cup$.

ANSWERS

EXERCISE 1 (p. 15)

1. 3 **2.** 3 **3.** 3 **4.** {1}; {2}; {3}; {1, 2}; {2, 3}; {1, 3}; { }; {1, 2, 3}

5. {0}; {1}; {2}; {0, 1}; {1, 2}; {0, 2}; { }; {0, 1, 2}

6. (*a*) {1, 3, 5 . . . 21}; (*b*) {2, 4, 6 . . . 20}; (*c*) {0, 1, 4, 9, 16, 25, 36, 49}; (*d*) {12, 15, 18, 21, 24, 27, 30}; (*e*) {11, 12, 13 . . . 19}; (*f*) ∅ or { }; (*g*) ∅; (*h*) {0, 1, 4, 9}; (*i*) {0, 1, 2, 3 . . . 10}

7. (*a*) The set of multiplies of 3 from 3 to 15 inclusive (*or* the set of multiples of 3 greater than 2 and less than 16). (*b*) The set of multiples of 10 from 10 to 50 inclusive. (*c*) The set of multiples of 4 from 4 to 28 inclusive. (*d*) The set of multiples of 4 from 4 to 24 inclusive. (*e*) The set of the squares of the first 9 counting numbers. (*f*) The set of multiples of 5 from 5 to 35 inclusive.

NOTE: *For all the above, other forms of wording are permissible, provided that they describe the sets just as clearly.*

8. C = {10, 20} (i) Yes (ii) Yes (iii) No

9. If C ⊂ B and B ⊂ A, then C ⊂ A.

EXERCISE 2 (p. 17)

1. Infinite **2.** Finite **3.** Infinite **4.** Infinite **5.** Finite **6.** Finite.

EXERCISE 3 (p. 21)

1. $\{8, 9, 10, 11, 12\}$ **2.** $\{8, 9, 10, 11, 12, 13\}$

3. $\{7, 8, 9, 10, 11, 12, 13\}$ **4.** $\{1, 2, 3, 4, \ldots\}$

5. $\{0, -1, -2, \ldots\}$ **6.** $\{5, 10, 15, \ldots\}$

7. $\{0, 5, 10, 15, \ldots\}$ **8.** $\{3, 2, 1, 0, -1, -2, -3\}$

9. $\{4, 3, 2, 1, 0, -1, -2, -3, -4\}$

10. $\{202, 204, 206, \ldots\}$
Sets 4, 5, 6, 7, and 10 are infinite.

11. $\{3x \mid x \text{ an integer}, 0 < x < 10\}$

12. $\{3x \mid x \text{ an integer}, 3 < x < 10\}$

13. $\{5x \mid x \text{ an integer}, 0 < x < 20\}$

14. $\{5x \mid x \text{ an integer}, x > 20\}$
Sets 11, 12, and 13 are finite. Set 14 is infinite.

15. $\{2x-1 \mid x \text{ a positive integer}\}$

16. $\{2x \mid x \text{ a positive integer}\}$
Sets 15 and 16 are infinite.

17. $\left\{\frac{1}{x} \mid x \text{ an integer}, 0 < x < 21\right\}$

18. $\{x \mid x \text{ integer}, \frac{1}{2} < x < \frac{3}{4}\}$

19. $\{x \mid x \text{ integer}, 100 < x < 10\}$
Sets 17, 18, and 19 are finite.

NOTE: The empty set $\emptyset$ may be written in set-builder notation thus: $\{x \mid x \neq x\}$

20. The sets comprise the numbers 3, 2, 1, 0 taken singly, in twos, and in threes, together with the set $\{3, 2, 1, 0\}$ and the empty set.

EXERCISE 4 (p. 31)

1. The set consisting of all the elements in a given universe which are not in A. A′.

2. Area outside A shaded.

3. $C = \{x \mid x \in A \text{ and } x \in B\} = A \cap B$.

4. Subsets must comprise the complete universe without overlapping.

5. (i) $\{2, 4, 6, 8\}$; (ii) $\{2, 3, 4, 6, 7, 8, 9, \}$; (iii) $\{1, 5\}$; (iv) $\{2, 3, 4, 6, 7, 8, 9\}$

6. $\{1, 3, 5, 6, 7\}$

7. (i) $\{17, 16, 15, 10, 9, 8\}$; (ii) $\{17, 16, 9, 8\}$; (iii) $\{14, 13, 12, 11\} = A$; (iv) = (i)

8. (i) $\{a, c\}$; (ii) $\{b, d, e, f\}$; (iii) $\{b, d, f\}$; (iv) $\{e, f\}$

9. (i) A and B disjoint; (ii) $A \subset B$; (iii) $B \subset A$.

10. (i) See figure 13; (ii) Figure 7 with A′ in place of B; (iii) No. $\emptyset$ has no elements, therefore the elements common to A and $\emptyset = \emptyset$.

11. (i) The set of all people who are both tall and dark. (ii) The set of all people who are *not both* tall and dark.

NOTE: The set $(A \cap B)'$ includes a person who is short, or a person who is blond, or a person who is short and blond.

EXERCISE 5 (p. 39)

1. (*a*) Those taking all three subjects; (*b*) those taking S and G, but not F; (*c*) those taking F and G, but not S; (*d*) those taking F and S, but not G; (*e*) those taking G, but not F or S; (*f*) those taking S, but not F or G; (*g*) those taking F, but not S or G; (*h*) those taking none of the three subjects.

3. 7 **4.** 152

5. (*a*) 5%; (*b*) 15%; (*c*) 22%; (*d*) 8%; (*e*) 8%; (*f*) 7%; (*g*) 12%

6. (*a*) 10%; (*b*) 35%; (*c*) 3%; (*d*) 3%

7. (*a*) 3%; (*b*) 5%; (*c*) 5%; (*d*) 0%; (*e*) 35%; (*f*) 10%; (*g*) 18%; (*h*) 24%

EXERCISE 6 (p. 43)

1. $\{5, 7\}$

2. (i) $\{1, 2, 7, 8, 9, 10\}$; (ii) $\{1, 2\}$; (iii) $\{1, 2\}$

4. (i) $A \cap B$; (ii) $B-A$, or $B \cap A'$; (iii) B'

EXERCISE 7 (p. 51)

1. (*a*) $\{3, 4\}$ (*b*) $\{3, 4\}$ (*c*) Yes (*d*) $\{1, 2, 3, 4, 5, 6\}$ (*e*) $\{1, 2, 3, 4, 5, 6\}$ (*f*) Yes (*g*) $\{5, 6, 7, 8, 9\}$ (*h*) $\{1, 2, 7, 8, 9\}$ (*i*) $\{1, 2, 5, 6, 7, 8, 9\}$ (*j*) $\{1, 2, 5, 6, 7, 8, 9\}$ (*k*) Yes (*l*) $A \cup A' =$ elements in A and elements not in $A = U$.
(*m*) A and A' are disjoint by definition.

2. (i) A and B shaded; (ii) A and B shaded; (iii) A and B shaded.

3. (i) No shading; (ii) Part common to both shaded; (iii) A shaded.

4. Whole universe shaded. **5.** A unshaded.

6. Whole figure shaded.

7. (*a*) Circles disjoint; (*b*) B is inside A; (*c*) A inside B; (*d*) B inside A; (*e*) A inside B.

8. (i) $A \cap B$; (ii) $A \cup B$; (iii) A'; (iv) B'; (v) B, or $A \cap B$, or $B \cap A$; (vi) $A-B$; (vii) $A' \cap B'$ or $(A \cup B)'$; (viii) $(A \cap B)'$, or $A' \cup B'$; (ix) A' (complement of A); (x) $(A \cap B) \cup (A \cap C)$; (xi) $A \cup B \cup C$; (xii) $A' \cap B \cap C$.

EXERCISE 8 (p. 64)

(*a*) $\{12\}$ (*b*) $\{3, -3\}$ (*c*) $\{5, -4\}$ (*d*) $\emptyset$ (*e*) $\left\{\frac{4}{3}, \frac{3}{2}\right\}$ (*f*) $\emptyset$ (*g*) $\left\{-\frac{5}{2}, 1\right\}$ (*h*) $\left\{\frac{3+\sqrt{44}}{5}, \frac{3-\sqrt{44}}{5}\right\}$ (*i*) $\emptyset$

ORAL EXERCISE, p. 66

(i) $\{2\}$ (ii) $\{-2, 2\}$ (iii) $\{x \mid x \text{ real, greater than } 1\}$
(iv) $\{x \mid x \text{ real, less than } 1\}$ (v) $\{x \mid x \text{ real, and } \neq 0\}$
(vi) $\{1, 2, 3, 4, 5\}$ (vii) $\{x \mid x \text{ real}, -1 < x < 3\}$
(viii) $\{-3, -2, -1, 0, 1, 2, 3, 4, 5\}$

EXERCISE 9 (p. 66)

1. Solution set $\{5, 6\}$ indicated by black dots.

2. Open circle at pt. 3. Heavy arrowed line to right.

3. $\{4, 5, 6\}$

4. Black circle at 3. Heavy arrowed line to right.

5. $\{2\}$ **6.** $\{4, 5\}$ **7.** $\{3, -3\}$ **8.** $\{3, -3\}$

9. Open circle at 5. Heavy arrowed line to right.

10. Open circle at 5. Heavy arrowed line to left.

EXERCISE 10 (p. 71)

1. (i) $\{6, 7, 8, 9\}$. Graph shows four isolated points. (ii) $\{5, 6, 7, 8, 9\}$. Graph shows five isolated points. (iii) $\{0, 1, 2, 3, 4\}$. Graph shows these five points. (iv) Sol. set as given. Graph: open circle at 5, heavy arrowed line to right. (v) Sol. set as given. Graph: black circle at 5, heavy arrowed line to right.

2. Graph of A: open circle at 9, heavy arrowed line to left. Graph of B: open circle at 3, heavy arrowed line to right.

3. $A \cap B$ = all real numbers > 3 and < 9; $A \cup B$ = all real numbers.

4. Graph consists of black circles at 3 and 7 and line between. Graph of $\{x \mid 3 \leqq x\}$ is black circle at 3 and arrowed line to right.

Graph of $\{x \leqq 7\}$ is black circle at 7 and arrowed line to left. The required intersection is the part of the number line in both graphs.

5. Black circles at 4 and -3.

6. Open circle at -3 with heavy line to left and open circle at $+4$ and heavy line to the right.

7. $\cup$ = (i) all points to the right of 4, and (ii) all points to the left of -3. This describes the graph in question 6.

8. First two sets are (i) all points to the *left* of 4, and (ii) all points to the *right* of -3. Therefore $\cap$ = all points between -3 and 4. This is the omitted section of the real number line in question 6, and so is the solution set required.

9. Graph includes all numbers greater than -4. But when x lies between 4 and -4, x^2 is *not* greater than 16.

10. When $x > 4$, $x^2 > 16$ and when $x < -4$, $x^2 > 16$.

11. $\{4, -3\}$. Two isolated points.

12. $\{x \mid x > 4$, or $x < -3\}$. Open circles at 4 and -3, heavy line to right of 4 and to the left of -3.

13. $\{x \mid -3 < x < 4\}$: open circles as in (12), with heavy line between.

14. $\{2\frac{1}{2}, -3\}$. Two isolated points.

15. $\{x \mid -3 < x < 2\frac{1}{2}\}$. Open circles at -3 and $2\frac{1}{2}$, with heavy line between.

16. $\{x \mid x < -3$, or $x > 2\frac{1}{2}\}$. Open circles as in (15), heavy line to the right of $2\frac{1}{2}$ and to the left of -3.

17. $\emptyset$; no mark on real number line. (Expression has no real factors.)

18. $\{x \mid x$ real$\}$; whole number line made heavy.

19. $\left\{-\frac{4}{3}, \frac{3}{2}\right\}$. Two isolated points.

20. $\left\{x \mid -\frac{4}{3} \leqq x \leqq \frac{3}{2}\right\}$. Closed circles at $-\frac{4}{3}$ and $\frac{3}{2}$, with heavy line between.

EXERCISE 11 (p. 76)

1. Double-hatched portion for each side is area common to circles A and C.

2. A and $(B \cap C)$ are disjoint; $(A \cap B)$ and C are disjoint.

EXERCISE 12 (p. 78)

1. For $A \cup (B \cap C)$, shade first the part common to B and C; this gives $(B \cap C)$.

The union of this set with A is all A and the part of the shaded portion not included in A.

Right side. $(A \cup B) \cap (A \cup C)$ is the double-hatched part got by first shading all of A and B vertically, and then all of A and C horizontally.

IMPORTANT. This result illustrates a rule of the algebra of sets that has no parallel in the ordinary algebra of numbers; because $a+(b \times c)$ is not generally equal to $(a+b) \times (a+c)$. However, $A \cup (B \cap C)$ *is* equal to $(A \cup B) \cap (A \cup C)$.

Thus not only do the associative, commutative, and distributive laws of the ordinary algebra of numbers hold for the algebra of sets, but we have an extra distributive law in the algebra of sets.

EXERCISE 13 (p. 78)

1. (*a*) 1st, 2nd, and 3rd year students. (*b*) 1st, 2nd, 3rd, and 4th year students. (*c*) 3rd and 4th year students in the opera. (*d*) 3rd year students in the opera and *all* 4th year students. (*e*) Tennis-playing students who are in the opera. (*f*) Same as (*e*). (*g*) Tennis-playing students who are *not* in the opera. (*h*) Students who are not in the opera and who don't play tennis. (*i*) Same as (*h*). (*j*) 1st year students in the opera. (*k*) 1st year students not in the opera. (*l*) All 1st year students ($= j \cup k$) $l = A \cap (M \cup M') = A \cap U = A$

2. (*a*) U; (*b*) $\emptyset$; (*c*) B; (*d*) A; (*e*) B; (*f*) A; (*g*) $\emptyset$; (*h*) $\emptyset$

3. (*a*) $\{5, 6, 7\}$; (*b*) $\{1, 2, 7\}$; (*c*) $\{3, 4\}$; (*d*) $\{1, 2, 3, 4, 5, 6\}$; (*e*) $\{3, 4, 5, 6, 7\}$; (*f*) $\{1, 2, 3, 4, 7\}$; (*g*) $\{5, 6\}$; (*h*) $\{1, 2\}$; (*i*) $\{1, 2, 5, 6, 7\}$; (*j*) $\{7\}$; (*k*) $\{7\}$

4. (*a*) A; (*b*) A; (*c*) $\emptyset$; (*d*) U; (*e*) A; (*f*) A′; (*g*) A; (*h*) A; (*i*) $\emptyset$; (*j*) U; (*k*) $A' \cap B'$; (*l*) $A' \cup B'$

5. (*a*) B shaded; (*b*) area common to B and C; (*c*) A; (*d*) shade B and C; (*e*) shade A and C; (*f*) shade all except B; (*g*) shading covers everything except area common to B and C; (*h*) A not shaded; (*i*) B and C not shaded.

6. and **7.** Diagrams.

8. $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, $\{c, a\}$, $\{a, b, c\}$. All the subsets of $D = \{a, b, c, d\}$ can be obtained from those of set C by including *d*, or not including *d*, with each of the 8 elements of C; this gives $2 \cdot 8 = 2^4$, and so on up to *n*.

9. (*a*) $I \cap R$; (*b*) $R \cap S$; (*c*) $\emptyset$

10. (i) True; (ii) false; (iii) false; (iv) true; (v) true; (vi) true; (vii) true; (viii) true.

EXERCISE 14 (p. 85)

1. Intersection = (2, 2); union = both lines in all cases.
2. $(1\frac{1}{2}, 1\frac{1}{2})$; **3.** (2, −1); **4.** (4, 1); **5.** $\emptyset$

EXERCISE 15 (p. 88)

1. B = the line itself and all points vertically over it; C = line and all points vertically below it.

2. D = straight line joining (−2, −1) and (3, 14): E = all points vertically over the line but the line itself excluded; F = all points vertically under the line (line excluded).

3. (3, 5)

4. (i) $\{(x, y) \mid y > 3, -1 \leqq x \leqq 4\}$;
(ii) $\{(x, y) \mid y < 0, -1 \leqq x \leqq 4\}$

5. Shaded area comprises all points above line (which is not included).

6. Shaded area comprises all points under line (which is not included).

7. Shaded area that to the left of the *y*-axis bounded by $y = x$ and $y = -x$. The lines themselves, however, are not included.

EXERCISE 16 (p. 90)

1. (2, 3); **2.** (−1, 4)

3. Area of *xy*-plane above both $y = x$ and $y = -x$, with lines excluded.

4. Area of *xy*-plane above both $y = x+4$ and $y = -x+1$, with lines excluded.

5. Area of plane above $y = x+4$ and *below* $y = -x+1$, with lines excluded.

6. As in question 5, but both lines are included.

7. (1, −1)

8. Area above both $y = 2x-3$ and $2y = -x-1$, and including both lines.

EXERCISE 17 (p. 94)

1. (*a*) 6; (*b*) 4; (*c*) 2; (*d*) 2; (*e*) 17; (*f*) 0

2. All true except (*a*) and (*d*).

3. Graph includes all numbers greater than -4 and less than $+4$, which are indicated on the real number line by open circles with heavy black line between.

4. The two isolated points, -1 and -7.

5. Graph includes all numbers greater than -1, and all numbers less than -7, which are indicated by open circles with heavy line to right of -1 and to left of -7.

6. Graph includes all numbers > -7 and < -1, which are indicated by open circles with heavy line from -7 to -1.

7. All real numbers except -1 and -7.

8. The first.

9. (*a*) -4; (*b*) $+4$

EXERCISE 18 (p. 99)

1. $(-2, 0)$, $(3, 0)$

2. (i) $\{(x, y) \mid y = x^2-x-6\}$; (ii) $\{(x, y) \mid y > x^2-x-6\}$; (iii) $\{(x, y) \mid y < x^2-x-6\}$

3. Curve is a parabola, passing through the origin, and symmetrical about the y-axis. Line passes through $(-12, 0)$ and $(0, 12)$. Required intersection is $(-3, 9)$ and $(4, 16)$.

4. Shade the portion of the interior of the curve cut off by the line and below it.

5. Shade the areas *above* the line, bounded by the curve but lying outside it. (Portion of curve above the straight line now indicated by a broken line.)

6. Two parabolas, meeting at the origin, and symmetrical, respectively, about the y- and $-y$-axes. Each is an image of the other.

7. A is a parabola, passing through origin, and symmetrical about y-axis; B also passes through the origin and is symmetrical about x-axis.

8. A rectangular hyperbola in 1st and 3rd quadrants. Axes are asymptotes. Curve in two parts, symmetrical about the origin.

9. As question 8, but in 2nd and 4th quadrants.

10. Graph $xy = 0$ consists of the two axes.
For $xy > 0$, 1st and 3rd quadrants shaded.

EXERCISE 19 (p. 113)

Nos. 1, 2, 3, 4, already done in the text.

5. $(A-B) = A \cap B'$
Let $x \in (A-B)$ $\therefore$ $x \in A$ and $x \notin B$, $\therefore$ $x \in (A \cap B')$

6. Since the distributive law
$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ has been proved for any three sets, it is true for their complements.
Therefore $(A' \cap B') \cup C' = (A' \cup C') \cap (B' \cup C')$
$(A \cup B)' \cup C' = (A \cap C)' \cap (B \cap C)'$
and $[(A \cup B) \cap C]' = [(A \cap C) \cup (B \cap C)]'$
Taking complements of both sides gives the required result $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. In other words: Intersection distributes over union.

EXERCISE 20 (p. 119)

1. (*a*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(*b*) $(A \cap \emptyset = \emptyset$ (*c*) $A \cup \emptyset = A$ (*d*) $A \cap (A \cup B) = A$
(*e*) $(A \cup B)' = A' \cap B'$

2. (*a*) A. Hint: $(A \cap B') \cup (A \cap B) = A \cap (B' \cup B)$
(*b*) $(A \cup B) \cap (A \cup C)$. Law 8*a* (p. 116)
(*c*) A′. (Hint: A′ ∩ is distributed over B′ ∪ B)
(*d*) A′. (Hint: Replace $(A \cup B)'$ by $A' \cap B'$ and proceed as in (*c*))
(*e*) $(A' \cup B) \cap (C \cup D)$—example 5 reversed

3. (*a*) A; (*b*) A; (*c*) A (dual of *b*); (*d*) U; (*e*) ∅ (dual of *d*); (*f*) A′ ∪ B. (Got by distributing B over the intersection)

EXERCISE 21 (p. 124)

No. 1.

	I	II	III	IV
	A	B	A ∪ B	B ∪ A
1.	∈	∈	∈	∈
2.	∈	—	∈	∈
3.	—	∈	∈	∈
4.	—	—	—	—

III and IV identical

No. 2.

	I	II	III	IV
	A	B	A ∩ B	A ∪ (A ∩ B)
1.	∈	∈	∈	∈
2.	∈	—	—	∈
3.	—	∈	—	—
4.	—	—	—	—

I and IV identical.

No. 3.

	I	II	III	IV	V	VI	VII
	A	B	A ∩ B	A′	B′	(A′ ∪ B′)	(A′ ∪ B′)′
1.	∈	∈	∈	—	—	—	∈
2.	∈	—	—	—	∈	∈	—
3.	—	∈	—	∈	—	∈	—
4.	—	—	—	∈	∈	∈	—

III and VII identical

No. 4. Dual of first example done in text.

No. 5.

	I	II	III	IV	V
Left Side	A	B	A ∩ B	(A ∩ B)′	A ∩ (A ∩ B)′
1.	∈	∈	∈	—	—
2.	∈	—	—	∈	∈
3.	—	∈	—	∈	—
4.	—	—	—	∈	—

Right Side

	VI	VII	VIII	IX
	A	B	B′	A ∩ B′
1.	∈	∈	—	—
2.	∈	—	∈	∈
3.	—	∈	—	—
4.	—	—	∈	—

V and IX identical.

EXERCISE 22 (p. 128)

(*a*) True in both: Nos. 1, 3, 8, 11, 14
(*b*) True only in Boolean algebra: 2, 4, 6, 7, 9, 10, 12, 13
(*c*) True only in ordinary algebra: 5

EXERCISE 23 (p. 128)

1. $A \cup \emptyset = A$ **2.** $A \cup U = U$
3. $A \cap U = A$ **4.** $A \cap A = A$
5. No sense in Boolean algebra, as $a \,.\, a = a$
6. $A \cup A' = U$ **7.** $A \cap A' = \emptyset$
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
10. No change
11. $A \cap \emptyset = \emptyset$ **12.** $\emptyset' = U$
13. $U \cup U = U$ **14.** $U \cup \emptyset = U$

EXERCISE 24 (p. 133)

1. 0 **2.** 1 **3.** 1 **4.** a **5.** a **6.** ab **7.** ab **8.** 0
9. $ab+a'b'$ **10.** a

EXERCISE 25 (p. 134)

1. $a(b+c)$ **2.** ab **3.** a **4.** ab **5.** ab
6. $(a+b)(c+d)$ **7.** $(A \cup B) \cap (C \cup D)$ **8.** $A \cap B$
9. $(A' \cup C) \cap (B' \cup D')$ **10.** U **11.** $(a+b)(a+c)$
12. $(a+b)(a+c)(a+d)$
13. $(a+c)(a+d)(a+f)(b+c)(b+d)(b+f)$
14. $(a+b)(a'+b')$ **15.** $(a+b')(a'+b)$

EXERCISE 26 (p. 137)

No. 7

L.H.S. $= [a'b+b'c+c'a]'$
$= (a'b)'.\,(b'c)'.\,(c'a)'$
$= (a+b')(b+c')(c+a') =$ R.H.S.

No. 10

L.H.S. $= abc+a'bc+ab'c+abc'$
$= abc+a'bc+abc+ab'c+abc+abc'$
$= bc(a+a')+ac(b+b')+ab(c+c')$
$= bc+ca+ab =$ R.H.S.

NOTE: $abc = abc+abc+abc$ (second line), and $a+a' = 1 = b+b' = c+c'$.

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